Journal of Mathematical Inequalities
Volume 16, Number 2 (2022), 609–628
doi:10.7153/jmi-2022-16-44

ON NEW SHARP BOUNDS FOR THE TOADER–QI MEAN INVOLVED IN THE MODIFIED BESSEL FUNCTIONS OF THE FIRST KIND

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Abstract. Let $A(a,b)$, $G(a,b)$, $L(a,b)$ and $TQ(a,b)$ be the arithmetic, geometric, logarithmic and Toader-Qi means of $a,b>0$ with $a\neq b$, respectively. Let $I_v(x)$ be the modified Bessel functions of the first kind of order v. We prove the double inequality

$$\sqrt{\frac{\sinh t}{t}U_q(t)} < I_0(t) < \sqrt{\frac{\sinh t}{t}U_p(t)}$$

holds for $t>0$, or equivalently,

$$\sqrt{L(a,b)\,\mathscr{U}_q(a,b)} < TQ(a,b) < \sqrt{L(a,b)\,\mathscr{U}_p(a,b)},$$

holds for $a,b>0$ with $a\neq b$, if and only if $p\geqslant 11/15$ and $0<q\leqslant 2/\pi$, where

$$\begin{aligned} U_p(t) &= p\cosh t - 4\left(p-\frac{2}{3}\right)\cosh\frac{t}{2} + 3p - \frac{5}{3}, \\ \mathscr{U}_p &= pA - 4\left(p-\frac{2}{3}\right)\sqrt{\frac{A+G}{2}G} + \left(3p-\frac{5}{3}\right)G. \end{aligned}$$

These improve some known results, in which $\sqrt{L\mathscr{U}_{2/\pi}}$ is the sharpest lower mean bound for TQ.

Mathematics subject classification (2020): Primary 39B62, 33B10; Secondary 26E60, 26D15.
Keywords and phrases: Modified Bessel function of the first kind, hyperbolic function, mean, inequalities.

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