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A DOUBLE INEQUALITY FOR THE APÉRY CONSTANT

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Abstract. A remarkable result which led to Apéry's proof of the irrationality of $\zeta(3)$ is given by the rapidly convergent series

$$\zeta(3)=\frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3\binom{2k}{k}}=\frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(k!)^2}{k^3(2k)!}.$$

Let

$$R_n=\zeta(3)-\frac{5}{2}\sum_{k=1}^{n}\frac{(-1)^{k-1}(k!)^2}{k^3(2k)!}$$

denote the remainder of the series. In this paper, we obtain an asymptotic expansion of $(-1)^nR_n$. Based on the obtained result, we establish the upper and lower bounds of $(-1)^nR_n$. As an application of the obtained bounds, we give an approximate value of $\zeta(3)$.

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