THE BOUNDS OF EIGENVALUE FOR COMPLEX
SINGULAR BOUNDARY VALUE PROBLEMS

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Abstract. The present paper deals with "the perturbation of Legendre eigenvalue problem" with limit-circle type non-oscillation endpoints. The dissipative operators in limit-circle case are studied. Lower bounds on the real parts of all eigenvalues and the upper bounds on the imaginary parts of the non-real eigenvalues for this eigenvalue problem associated to a special separated boundary condition (see the below in (**??**)) are obtained through a new method, partly inspired by the estimates obtained in Sun and Qi (Proc. Roy. Soc. Edinburgh A, 150:2607-2619, 2020).

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