# THE SHARP BOUND OF THE THIRD HANKEL
# DETERMINANT FOR CONVEX FUNCTIONS OF ORDER $-1/2$

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*Abstract.* We prove the sharp inequality $|H_{3,1}(f)| \leqslant 1/16$ for the third Hankel determinant $H_{3,1}(f)$ for convex functions of order $-1/2$ i.e., functions $f$ analytic in $z \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ with $a_n := f^{(n)}(0)/n!$, $n \in \mathbb{N}$, $a_1 := 1$, such that

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > -\frac{1}{2}, \quad z \in \mathbb{D},$$

thus proving a recent conjecture.

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