

NORM-PARALLELISM OF HILBERT SPACE OPERATORS AND THE DAVIS–WIELANDT BEREZIN NUMBER

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Abstract. In this work, the concept of the Davis-Wielandt Berezin number is introduced. Some upper and lower bounds for the Davis-Wielandt Berezin number are introduced. A connection between norm-parallelism to the identity operator and an equality condition for the Davis-Wielandt Berezin number are also discussed. Some bounds for the Davis-Wielandt Berezin number for $n \times n$ operator matrices are established.

Mathematics subject classification (2020): Primary 47A30, 47A12; Secondary 47A63, 47L05.

Keywords and phrases: Davis-Wielandt radius, Davis-Wielandt shell, Berezin symbol, $n \times n$ operator matrix.

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