Journal of Mathematical Inequalities
Volume 17, Number 2 (2023), 459–480 doi:10.7153/jmi-2023-17-31

ON APPROXIMATELY CONVEX AND AFFINE FUNCTIONS

ANGSHUMAN R. GOSWAMI AND ZSOLT PÁLES

Abstract. A real valued function f defined on a real open interval I is called Φ-convex if, for all $x,y \in I$, $t \in [0,1]$ it satisfies

$$f(tx+(1-t)y) \leqslant tf(x)+(1-t)f(y)+t\Phi\big((1-t)|x-y|\big)+(1-t)\Phi\big(t|x-y|\big),$$

where $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonnegative error function. If f and $-f$ are simultaneously Φ-convex, then f is said to be a Φ-affine function. In the main results of the paper, we describe the structural and inclusion properties of these two classes. We characterize these two classes of functions and investigate their relationship with approximately monotone and approximately-Hölder functions. We also introduce a subclass of error functions that enjoys the so-called Γ property and we show that the error function which is the most optimal for a Φ-convex function has to belong to this subclass. The properties of this subclass of error function are investigated as well. Then we offer two formulas for the lower Φ-convex envelop. Besides, a sandwich type theorem is also added.

Mathematics subject classification (2020): Primary 26A48; Secondary 26A12, 26A16, 26A45, 39B72, 39C05.

Keywords and phrases: Φ-monotone function, Φ-Hölder function, Φ-monotone envelope, Φ-Hölder envelope, Ostrowski-type inequality, Hermite–Hadamard-type inequality.

REFERENCES

[1] M. ADAMEK, *On F-Convex Functions*, J. Convex Anal., 28 (1), 2021.
[2] K. BARON, J. MATKOWSKI, AND K. NIKODEM, *A sandwich with convexity*, Math. Pannon., 5 (1): 139–144, 1994.
[3] Z. BOROS AND T. NAGY, *Approximately convex functions*, Ann. Univ. Sci. Budapest. Sect. Comput., 40: 143–150, 2013.
[4] P. BURAI AND A. HÁZY, *On approximately h-convex functions*, J. Convex Anal., 18 (2): 447–454, 2011.
[5] P. BURAI, A. HÁZY, AND T. JUHÁSZ, *On approximately Breckner s-convex functions*, Control Cybernet., 40 (1): 91–99, 2011.
[6] S. J. DILWORTH, R. HOWARD, AND J. W. ROBERTS, *Extremal approximately convex functions and estimating the size of convex hulls*, Adv. Math., 148 (1): 1–43, 1999.
[7] S. J. DILWORTH, R. HOWARD, AND J. W. ROBERTS, *On the size of approximately convex sets in normed spaces*, Studia Math., 140 (3): 213–241, 2000.
[8] S. J. DILWORTH, R. HOWARD, AND J. W. ROBERTS, *Extremal approximately convex functions and the best constants in a theorem of Hyers and Ulam*, Adv. Math., 172 (1): 1–14, 2002.
[9] R. GER, *Almost approximately convex functions*, Math. Slovaca, 38 (1): 61–78, 1988.
[10] A. GILÁNYI, C. GONZÁLEZ, K. NIKODEM, AND ZS. PÁLES, *Bernstein-Doetsch type theorems with Tabor type error terms for set-valued maps*, Set-Valued Var. Anal., 25 (2): 441–462, 2017.
[11] C. GONZÁLEZ, K. NIKODEM, ZS. PÁLES, AND G. ROA, *Bernstein-Doetsch type theorems for set-valued maps of strongly and approximately convex and concave type*, Publ. Math. Debrecen, 84 (1–2): 229–252, 2014.
[12] A. GOSWAMI AND ZS. PÁLES, *On approximately monotone and approximately Hölder functions*, Per. Math. Hungar., 81 (1): 65–87, 2020.

Paper JMI-17-31

[13] A. GOSWAMI AND ZS. PÁLES, *Characterization of approximately monotone and approximately Hölder functions*, Math. Inequal. Appl., 24 (1): 247–264, 2021.

[14] J. W. GREEN, *Approximately convex functions*, Duke Math. J., 19: 499–504, 1952.

[15] D. H. HYERS AND S. M. ULAM, *Approximately convex functions*, Proc. Amer. Math. Soc., 3: 821–828, 1952.

[16] A. HÁZY, *On approximate t-convexity*, Math. Inequal. Appl., 8 (3): 389–402, 2005.

[17] A. HÁZY, *Bernstein-Doetsch type results for generalized approximately convex functions*, In Proc. MicroCAD 2010 Int. Sci. Conf., volume H, page 27–32, 2010.

[18] A. HÁZY AND ZS. PÁLES, *On approximately midconvex functions*, Bull. London Math. Soc., 36 (3): 339–350, 2004.

[19] A. HÁZY AND ZS. PÁLES, *On approximately t-convex functions*, Publ. Math. Debrecen, 66 (3–4): 489–501, 2005.

[20] W. JARCZYK AND M. LACZKOVICH, *Convexity on abelian groups*, J. Convex Anal., 16 (1): 33–48, 2009.

[21] W. JARCZYK AND M. LACZKOVICH, *Almost convex functions on locally compact abelian groups*, Math. Inequal. Appl., 13 (1): 217–225, 2010.

[22] J. KRZYSZKOWSKI, *Approximately generalized convex functions*, Math. Pannon., 12 (1): 93–104, 2001.

[23] M. LACZKOVICH, *The local stability of convexity, affinity and of the Jensen equation*, Aequationes Math., 58: 135–142, 1999.

[24] J. MAKÓ, *A new proof of the approximate convexity of the Takagi function*, Acta Math. Hungar., 151 (2): 456–461, 2017.

[25] J. MAKÓ AND A. HÁZY, *On approximate Hermite-Hadamard type inequalities*, J. Convex Anal., 24 (2): 349–363, 2017.

[26] J. MAKÓ AND ZS. PÁLES, *Approximate convexity of Takagi type functions*, J. Math. Anal. Appl., 369 (2): 545–554, 2010.

[27] J. MAKÓ AND ZS. PÁLES, *Strengthening of strong and approximate convexity*, Acta Math. Hungar., 132 (1–2): 78–91, 2011.

[28] J. MAKÓ AND ZS. PÁLES, *Implications between approximate convexity properties and approximate Hermite-Hadamard inequalities*, Cent. Eur. J. Math., 10 (3): 1017–1041, 2012.

[29] J. MAKÓ AND ZS. PÁLES, *Korovkin type theorems and approximate Hermite-Hadamard inequalities*, J. Approx. Theory, 164 (8): 1111–1142, 2012.

[30] J. MAKÓ AND ZS. PÁLES, *On ϕ-convexity*, Publ. Math. Debrecen, 80 (1–2): 107–126, 2012.

[31] J. MAKÓ AND ZS. PÁLES, *Approximate Hermite-Hadamard type inequalities for approximately convex functions*, Math. Inequal. Appl., 16 (2): 507–526, 2013.

[32] J. MAKÓ AND ZS. PÁLES, *On approximately convex Takagi type functions*, Proc. Amer. Math. Soc., 141 (6): 2069–2080, 2013.

[33] J. MROWIEC, *Remark on approximately Jensen-convex functions*, C. R. Math. Acad. Sci. Soc. R. Can., 23 (1): 16–21, 2001.

[34] J. MROWIEC, JA. TABOR, AND JÓ. TABOR *Approximately midconvex functions*, In *Inequalities and applications*, volume 157 of Internat. Ser. Numer. Math., page 261–265, Birkhäuser, Basel, 2009.

[35] C. T. NG AND K. NIKODEM, *On approximately convex functions*, Proc. Amer. Math. Soc., 118 (1): 103–108, 1993.

[36] K. NIKODEM AND ZS. PÁLES, *On approximately Jensen-convex and Wright-convex functions*, C. R. Math. Rep. Acad. Sci. Canada, 23 (4): 141–147, 2001.

[37] K. NIKODEM AND ZS. PÁLES, *Generalized convexity and separation theorems*, J. Convex Anal., 14 (2): 239–248, 2007.

[38] ZS. PÁLES, *Separation by approximately convex functions*, In D. Gronau and L. Reich, editors, *Contributions to the theory of functional equations, III* (Graz, 2000), volume 344 of Grazer Math. Ber., page 43–50, Karl-Franzens-Univ. Graz, Graz, 2001.

[39] ZS. PÁLES, *On approximately convex functions*, Proc. Amer. Math. Soc., 131 (1): 243–252, 2003.

[40] S. ROLEWICZ, *On γ-paraconvex multifunctions*, Math. Japon., 24 (3): 293–300, 1979.

[41] P. SPUREK AND JA. TABOR, *Numerical verification of condition for approximately midconvex functions*, Aequationes Math., 2012.

[42] JA. TABOR, JÓ. TABOR, AND M. ŻOŁDAK, *Approximately convex functions on topological vector spaces*, Publ. Math. Debrecen, 77: 115–123, 2010.

[43] JA. TABOR, JÓ. TABOR, AND M. ŻOŁDAK, *Optimality estimations for approximately midconvex functions*, Aequationes Math., 80: 227–237, 2010.

Journal of Mathematical Inequalities
www.ele-math.com
jmi@ele-math.com