

ASYMPTOTIC EXPANSION FOR GENERALIZED MATHIEU SERIES

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Abstract. The main result of this paper presents a series expansion of the generalized Mathieu series $S_m(r)$ with computable coefficients. This generalizes Elbert's work from $m = 1$ to arbitrary $m \in \mathbb{Z}_{>0}$, and provides a method for calculating the asymptotic formula of $S_m(r)$ within a given error. Moreover, this paper revises an integral representation for $S_m(r)$ given by Cerone and Lenard.

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