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IMPROVEMENTS OF A–NUMERICAL RADIUS FOR SEMI–HILBERTIAN SPACE OPERATORS

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Abstract. Let A be a bounded positive operator on a complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$. The semi-product $\langle x,y\rangle_A := \langle Ax,y\rangle, x,y\in H$, induces a semi-norm $\|\cdot\|_A$ on H. Let $\omega_A(T)$ and $\|T\|_A$ denote the A-numerical radius and the A-operator semi-norm of an operator T in semi-Hilbertian space $(H,\langle\cdot,\cdot\rangle_A)$, respectively. In this paper, some new bounds for the A-numerical radius of operators in semi-inner product space induced by A are derived. In particular, for $T\in\mathcal{B}_A(H)$ and $\alpha\geqslant 0$, we prove that

$$\omega_A^4(T)\leqslant\frac{1+2\alpha}{16(1+\alpha)}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2+\frac{3+2\alpha}{8(1+\alpha)}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A\omega_A(T^2)$$

and

$$\omega_A^4(T)\leqslant\frac{1+2\alpha}{8(1+\alpha)}\|T^{\sharp_A}T+TT^{\sharp_A}\|_A^2+\frac{1}{2(1+\alpha)}\omega_A^2(T^2).$$

It is worth noting that our results improve the existing A-numerical radius inequalities. Further, we also give a refinement inequality of A-operator semi-norm triangle inequality.

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REFERENCES

[1] M. AL-DOLATA, I. JARADAT, *A refinement of the Cauchy-Schwarz inequality accompanied by new numerical radius upper bounds*, Filomat, 2023, **37** (3): 971–977.
[2] M. L. ARIAS, G. CORACH, M. C. GONZALEZ, *Lifting properties in operator ranges*, Acta Mathematical Scientia, 2009, **75** (3): 635–653.
[3] M. L. ARIAS, G. CORACH, M. C. GONZALEZ, *Metric properties of projections in semi-Hilbertian spaces*, Integral Equations and Operator Theory, 2008, **62** (1): 11–28.
[4] M. L. ARIAS, G. CORACH, M. C. GONZALEZ, *Partial isometries in semi-Hilbertian spaces*, Linear Algebra and its Applications, 2008, **428** (7): 1460–1475.
[5] J. S. AUJLA, F. C. SILVA, *Weak majorization inequalities and convex functions*, Linear Algebra and its Applications, 2003, **369**: 217–233.
[6] H. BAKLOUTI, K. FEKI, O. A. M. SID, *Joint normality of operators in semi-Hilbertian spaces*, Linear and Multilinear Algebra, 2020, **68** (4): 845–866.
[7] P. BHUNIA, K. PAUL, R. K. NAYAK, *On inequalities for A-numerical radius of operators*, Electronic Journal of Linear Algebra, 2020, **36** (1): 143–157.
[8] P. BHUNIA, K. FEKI, K. PAUL, *Numerical radius parallelism and orthogonality of semi-Hilbertian space operators and its applications*, Bulletin of the Iranian Mathematical Society, 2021, **47** (1): 435–457.
[9] P. BHUNIA, R. K. NAYAK, K. PAUL, *Refinement of seminorm and numerical radius inequalities of semi-Hilbertian space operators*, Mathematica Slovaca, 2022, **72** (4): 969–976.
[10] C. CONDE, K. FEKI, *Some numerical radius inequality for several semi-Hilbert space operators*, Linear and Multilinear Algebra, 2023, **71** (6): 1054–1071.
[11] K. DAVIDSON, S. C. POWER, *Best approximation in C^*-algebras*, Journal Fur Die Reine Und Angewandte Mathematik, 1986, **368** (1): 43–62.

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[12] R. G. DOUGLAS, *On majorization, factorization and range inclusion of operators in Hilbert space*, Proceeding of the Ameriacan Mathematical Socociety, 1966, **17** (1): 413–416.

[13] M. FAGHIH-AHMADI, F. GORJIZADEH, *A-numerical radius of A-normal operators in semi-Hilbertian spaces*, Italian Journal of Pure and Applied Mathematics, 2016, **36** (1): 73–78.

[14] K. FEKI, *A note on the A-numerical radius of operators in semi-Hilbert spaces*, Archiv der Mathematik, 2020, **115** (5): 535–544.

[15] K. FEKI, *Some numerical radius inequalities for semi-Hilbert space operators*, Journal of the Korean Mathematical Society, 2021, **58** (6): 1385–1405.

[16] K. FEKI, *Spectral radius of semi-Hilbertian space operators and its applications*, Annals of Functional Analysis, 2020, **11** (1): 929–946.

[17] K. FEKI, *Spectral radius of semi-Hilbertian space operators and its applications*, Annals of Functional Analysis, 2020, **11** (4): 929–946.

[18] M. C. GONZALEZ, *Operator norm inequalities in semi-Hilbertian spaces*, Linear Algebra and its Applications, 2011, **434** (1): 370–378.

[19] M. GUESBA, *A-numerical radius inequalities via Dragomir inequalities and related results*, Novi Sad Journal Mathematics, 2023, https://doi.org/10.30755/NSJOM.15867.

[20] M. GUESBA, *Some generalizations of A-numerical radius inequalities for semi-Hilbert space operators*, Bollettino dell'Unione Matematica Italiana, 2021, **14**: 681–692.

[21] M. GUESBA, P. BHUNIA, K. PAUL, *A-numerical radius inequalities and A-translatable radii of semi-Hilbert space operators*, Filomat, 2023, **37** (11): 3443–3456.

[22] B. HAMADI, K. FEKI, O. A. M. SID AHMED, *Joint numerical ranges of operators in semi-Hilbertian spaces*, Linear Algebra and its Applications, 2018, **555** (1): 266–284.

[23] M. S. MOSLEHIAN, Q. XU, A. ZAMANI, *Seminorm and numerical radius inequalities of operators in semi-Hilbertian spaces*, Linear Algebra and its Applications, 2020, **591**: 299–321.

[24] H. QIAO, G. HAI, E. BAI, *A-numerical radius and A-norm inequalities for semi-Hilbertian space operators*, Linear and Multilinear Algebra, 2022, **70** (21): 6891–6907.

[25] H. QIAO, G. HAI, E. BAI, *Some refinements of numerical radius inequalities for 2×2 operator matrices*, Journal of Mathematical Inequalities, 2022, **16** (2): 425–444.

[26] A. SADDI, *A-Normal operators in Semi-Hilbertian spaces*, The Australian Journal of Mathematical Analysis and Applications, 2012, **9** (1): 1–12.

[27] Q. X. XU, Z. M. YE, A. ZAMANI, *Some upper bounds for the $\mathbb{A}$-numerical radius of 2×2 block matrices*, Advances in Operator Theory, 2021, **6** (1): 1–13.

[28] A. ZAMANI, *A-numerical radius inequalities for semi-Hilbertian space operators*, Linear Algebra and its Applications, 2019, **578**: 159–183.