

HÖLDER'S INEQUALITIES AND MULTILINEAR SINGULAR INTEGRALS ON GENERALIZED ORLICZ SPACES

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Abstract. Let $\varphi_j \in \Phi_w(\mathbb{R}^n)$, $j = 1, \dots, m$, be generalized Orlicz functions. We obtain a general version of the Hölder inequality on the generalized Orlicz spaces

$$\left\| \prod_{j=1}^m f_j \right\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)} \leq C \prod_{j=1}^m \|f_j\|_{L^{\varphi_j(\cdot)}(\mathbb{R}^n)},$$

where $\varphi = (\prod_{j=1}^m \varphi_j^{-1})^{-1}$. If every φ_j satisfies the conditions, (A0), (A1), (A2), (aInc) $_{p_j}$ and (aDec) $_{q_j}$ with $1 < p_j, q_j < \infty$, then the multilinear sparse operators and multilinear Calderón–Zygmund operators are bounded from $L^{\varphi_1(\cdot)}(\mathbb{R}^n) \times \dots \times L^{\varphi_m(\cdot)}(\mathbb{R}^n)$ to $L^{\varphi(\cdot)}(\mathbb{R}^n)$. We also establish the boundedness of the multilinear fractional integral operators over the generalized Orlicz spaces. These results are also new for classical Orlicz spaces as the special case.

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