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PARTIAL EIGENVALUES FOR BLOCK MATRICES

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Abstract. In this paper, we define extensions of the classical eigenvalues of the matrix $A \in \mathbb{M}_m(\mathbb{C})$. These extensions are eigenvalues matrices for the block matrix $A \in \mathbb{M}_m(\mathbb{M}_n)$, where $\mathbb{M}_m(\mathbb{M}_n)$ is the set of all $m \times m$ block complex matrices with each block in $\mathbb{M}_n(\mathbb{C})$, they are

$$\lambda_h^{(1)}(A) = [\lambda_h(G_{l,k})]_{l,k=1}^n, \text{ for } h = 1,2,\cdots,m$$

and

$$\lambda_h^{(2)}(A) = [\lambda_h(A_{i,j})]_{i,j=1}^m, \text{ for } h = 1,2,\cdots,n.$$

Among other equalities and inequalities, we prove equalities which relate our new definitions with $\operatorname{tr}_1(A)$ and $\operatorname{tr}_2(A)$ as follows,

$$\operatorname{tr}_1(A) = \sum_{h=1}^{m} \lambda_h^{(1)}(A)$$

and

$$\operatorname{tr}_2(A) = \sum_{h=1}^{n} \lambda_h^{(2)}(A).$$

These relations are extensions of the classical relation $\operatorname{tr}(A) = \sum_{h=1}^{m} \lambda_h(A)$, where $A \in \mathbb{M}_m(\mathbb{C})$. Several new relations and properties of our new definitions are also given.

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