

WILKER'S, CUSA–HUYGENS', AND FINK–MORTICI'S TYPE INEQUALITIES FOR PARABOLIC TRIGONOMETRIC FUNCTIONS

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Abstract. Parabolic trigonometric functions (PTF) have been recently defined as functions of an area lying on a parabolic circle. In this paper, we begin with an investigation into PTF's properties as these functions still need to be sufficiently understood. We find a precise formulation of Wilker's inequality together with Fink-Mortici's type inequality for PTF. Furthermore, we conjecture the proper form of Cusa-Huygens' inequality for PTF and find both upper and lower bounds on this inequality.

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