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INEQUALITIES FOR TWO POWER SERIES OF NONCOMMUTATIVE OPERATORS IN HILBERT SPACES WITH APPLICATIONS TO NUMERICAL RADIUS

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Abstract. Let H be a complex Hilbert space. We consider the power series with complex coefficients $f(z) := \sum_{k=0}^{\infty} a_k z^k$ with $a_k \in \mathbb{C}$ for $k \in \mathbb{N} := \{0,1,\dots\}$. Suppose that this power series is convergent on the open disk $D(0,R) := \{z \in \mathbb{C} | \, z < R\}$. We define $f_a(z) := \sum_{k=0}^{\infty} |a_k| z^k$ which has the same radius of convergence R. Assume that the power series $f(z) = \sum_{i=0}^{\infty} a_i z^i$ is convergent on the open disk $D(0,R_1)$ and $g(z) = \sum_{i=0}^{\infty} b_i z^i$ is convergent on $D(0,R_2)$ and A, B, C, D be operators in $B(H)$ with $\|A\|^{1/2}, \|A\| < R_1$ and $\|B\|^{1/2}$, $\|B\| < R_2$. In this paper we show among others that

$$|\langle D^* A f(A) g(B) BCx, y\rangle| \leqslant \|A\|^{\alpha} \|B\|^{1-\alpha} f_a\left(\|A\|^{1/2}\right) g_a\left(\|B\|^{1/2}\right) \times \left\langle \left||B|^{\alpha} C\right|^2 x, x\right\rangle^{1/2} \left\langle \left||A^*|^{1-\alpha} D\right|^2 y, y\right\rangle^{1/2}$$

for all $\alpha \in [0,1]$ and x, $y \in H$. Application for norm and numerical radius inequalities for the composite operator $D^* A f(A) g(B) BC$ are provided. Some examples for fundamental power series are also given.

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