

MONOTONICITY PROPERTIES OF THE GAUSSIAN HYPERGEOMETRIC FUNCTIONS WITH APPLICATIONS

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Abstract. This paper shows the monotonicity properties of combined functions involving the Gaussian hypergeometric function with certain conditions satisfied by their parameters. These results have important applications in the theories of quasiconformal mappings and Ramanujan's modular equations. By these results, several inequalities for the complete elliptic integrals and the generalized elliptic integrals, the Grötzsch ring function and the generalized Grötzsch ring function, the solutions of Ramanujan's modular equation and Ramanujan's generalized modular equation are obtained.

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