

## MEAN-TYPE INEQUALITIES FOR THE NUMERICAL RADIUS AND THE OPERATOR NORM

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*Abstract.* In this paper, utilizing the Hadamard product of matrices, we show several new bounds for the numerical radius in a way that extends some known bounds for the operator norm. However, the presented results treat special cases to overcome the general case, invalid for the numerical radius. As a consequence of our discussion, we find relations between the numerical radii of the Aluthge and Duggal transformations. Then, we show some bounds for the product of three Hilbert space operators, and some mean-like terms are treated using operator matrices techniques.

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### REFERENCES

- [1] A. ABU-OMAR, F. KITTANEH, *Upper and lower bounds for the numerical radius with an application to involution operators*, Rocky Mountain J. Math., **45**(4) (2015), 1055–1065.
- [2] A. ALUTHGE, *On  $p$ -hyponormal operators for  $0 < p < 1$* , Integral Equ. Oper. Theory., **13** (1990), 307–315.
- [3] C. BENHIDA, M. CHŌ, E. KO, AND J. E. LEE, *On the generalized mean transforms of complex symmetric operators*, Banach J. Math. Anal., **14** (2020), 842–855.
- [4] R. BHATIA, *Matrix Analysis*, Springer-Verlag, New York, 1997.
- [5] S. S. DRAGOMIR, *Some inequalities for the norm and the numerical radius of linear operators in Hilbert spaces*, Tamkang J. Math., **39** (1) (2008), 1–7.
- [6] C. FOIAS, I. JUNG, E. KO, AND C. PEARCY, *Complete contractivity of maps associated with the Aluthge and Duggal transformations*, Pacific J. Math., **209** (2003), 249–259.
- [7] T. FURUTA, *Invitation to Linear Operators*, Taylor and Francis, London, 2001.
- [8] I. H. GÜMÜŞ, H. R. MORADI, M. SABABHEH, *On positive and positive partial transpose matrices*, Electron. J. Linear Algebra., **38** (2022), 792–802.
- [9] K. E. GUSTAFSON, D. K. M. RAO, *Numerical Range. The Field of Values of Linear Operators and Matrices*, Springer-Verlag, New York, 1997.
- [10] M. HASSANI, M. E. OMIĐVAR, AND H. R. MORADI, *New estimates on numerical radius and operator norm of Hilbert space operators*, Tokyo J. Math., **44** (2) (2021), 439–449.
- [11] Z. HEYDARBEGYI, M. SABABHEH AND H. R. MORADI, *A convex treatment of numerical radius inequalities*, Czechoslovak Math. J. **72** (147) (2022), 601–614.
- [12] O. HIRZALLAH, F. KITTANEH, K. SHEBRAWI, *Numerical radius inequalities for commutators of Hilbert space operators*, Numer Funct Anal Optim., **32** (2011) 739–749.
- [13] O. HIRZALLAH, F. KITTANEH, AND K. SHEBRAWI, *Numerical radius inequalities for certain  $2 \times 2$  operator matrices*, Integr. Equ. Oper. Theory., **71** (2011), 129–147.
- [14] R. KAUR, M. S. MOSLEHIAN, M. SINGH, AND C. CONDE, *Further refinements of the Heinz inequality*, Linear Algebra Appl., **447** (2014), 26–37.
- [15] F. KITTANEH, *Notes on some inequalities for Hilbert space operators*, Publ. Res. Inst. Math. Sci., **24** (1988), 283–293.
- [16] F. KITTANEH, *A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix*, Studia Math., **158** (2003), 11–17.

- [17] F. KITTANEH, *Numerical radius inequalities for Hilbert space operators*, *Studia Math.*, **168** (2005), 73–80.
- [18] F. KITTANEH, H. R. MORADI, AND M. SABABHEH, *Sharper bounds for the numerical radius*, *Linear Multilinear Algebra*, <https://doi.org/10.1080/03081087.2023.2177248>.
- [19] S. LEE, W. LEE, AND J. YOON, *The mean transform of bounded linear operators*, *J. Math. Anal. Appl.*, **410** (2014), 70–81.
- [20] M. SABABHEH, *Numerical radius inequalities via convexity*, *Linear Algebra Appl.*, **549** (2018), 67–78.
- [21] M. SABABHEH, *Heinz-type numerical radii inequalities*, *Linear multilinear algebra.*, **67** (5) (2019), 953–964.
- [22] M. SINGH, J. S. AUJLA, AND H. L. VASUDEVA, *Inequalities for Hadamard product and unitarily invariant norms of matrices*, *Linear Multilinear Algebra.*, **48** (2001), 247–262.
- [23] A. SHEIKHHOSSEINI, *An arithmetic-geometric mean inequality related to numerical radius of matrices*, *Konuralp J. Math.*, **5** (1) (2017), 85–91.
- [24] S. SHEYBANI, M. SABABHEH, AND H. R. MORADI, *Weighted inequalities for the numerical radius*, *Vietnam J. Math.*, **51** (2023), 363–377.
- [25] T. YAMAZAKI, *On upper and lower bounds of the numerical radius and an equality condition*, *Studia Math.*, **178** (2007), 83–89.
- [26] X. ZHAN, *Inequalities for unitarily invariant norms*, *SIAM J. Matrix Anal. Appl.*, **20** (2) (1998), 466–470.