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# TWO–STEP MINIMIZATION APPROACH TO SOBOLEV–TYPE INEQUALITY WITH BOUNDED POTENTIAL IN 1D

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*Abstract.* We present a new method to determine the best constant of the Sobolev-type embedding in one dimension with a norm including a bounded inhomogeneous potential term. This problem is closely connected to the Green function of the Schrödinger operator with inhomogeneous potential. A minimization problem of a Rayleigh-type quotient in a Sobolev space gives the best constant of the Sobolev embedding. We decompose the minimization problem into two sub-minimization problems and show that the Green function provides the minimizer of the first minimization problem. Then, it enables us to derive a new precise estimate of the best constant and function for inhomogeneous bounded potential cases. As applications, we give some examples of the inhomogeneous potential whose best constant and function of the Sobolev-type embedding are explicitly determined.

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