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ON AN OPEN PROBLEM CONCERNING THE RECIPROCAL SUM RELATED TO THE RIEMANN ZETA–FUNCTION

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Abstract. In 2016, Lin studied the computational problem of the reciprocal sum related to the Riemann zeta function. More precisely, the author proved that, for any positive integer n,

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{k^2} \right)^{-1} \right\rfloor = n-1 \quad \text{and} \quad \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{k^3} \right)^{-1} \right\rfloor = 2n(n-1),$$

where $\lfloor x \rfloor$ is the floor function, that is, it denotes the greatest integer less than or equal to x. At the same time, Lin also proposed the following open problem: Whether there exists an explicit computational formula for $\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{k^s} \right)^{-1} \right\rfloor$, where s is an integer with $s \geqslant 4$. In this paper, we present the asymptotic expansion of $\left(\sum_{k=n}^{\infty} 1/k^{j+1} \right)^{-1}$ in terms of $1/n$. Based on this expansion, we answer the open problem of Lin for $s=4$ and $s=5$. Using our method and Maple software one can study the open problem of Lin for the cases $s \geqslant 6$.

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