

A LOWER BOUND FOR THE BETA FUNCTION

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Abstract. We establish a novel lower bound for Euler's beta function $B(x, y)$, asserting that the inequality

$$B(x, y) > \frac{x+y}{xy} \left(1 - \frac{2xy}{x+y+1} \right)$$

holds for all $(x, y) \in (0, 1] \times (0, 1]$. This bound refines the lower bound derived by P. Ivády [12, Theorem, (3.2)] in the case where $0 < x + y < 1$.

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