

HERMITE–HADAMARD AND GRADIENT INEQUALITIES FOR CONVEX FUNCTIONS

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Abstract. The Hermite-Hadamard inequality is one of the most exciting and applicable inequalities that govern convex functions. In this paper, we investigate convex and differentiable convex functions and present several new orderings among such functions. The results include refinements of the well-known gradient inequality for convex functions, a delicate refinement of the celebrated Hermite-Hadamard inequality, and many other interesting consequences.

As applications of the obtained results, we present some inequalities for scalar means, operator means, relative and Shannon entropies for scalars and operators.

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