

## THREE EQUIVALENT DISTANCES ABOUT SINGULAR SUBSPACES

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**Abstract.** Cai and Zhang proved that three distances between singular subspaces under spectral and Frobenius norms are equivalent (T. T. Cai and A. Zhang, *The Annals of Statistics*, 2018, **46** (1), 60–89). We show that these distances remain equivalent under any unitarily invariant norm. The constants in Theorems ??–?? (Cai and Zhang’s bounds) can be made sharper in our results (Theorems ??–??). Furthermore, our proof methods for Theorems ?? which extend Cai and Zhang’s results (Theorem ??) are significantly simpler.

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