

ON THE δ -NUMERICAL RADIUS AND THE δ -CRAWFORD NUMBER FUNCTIONS OF DIRECT SUM OF HILBERT SPACE OPERATORS

ZAMEDDIN I. ISMAILOV AND PEMBE IPEK AL*

Abstract. In this work, some important properties of the δ -numerical radius and the δ -Crawford number functions of bounded linear operators on Hilbert spaces are investigated. Then, the relationships between the δ -numerical radius and the δ -Crawford number functions of the direct sum of Hilbert space operators and those of their coordinate operators are explored. Furthermore, some generalization formulas for lower and upper bounds of the δ -numerical radius and the δ -Crawford number functions of such direct sums are obtained. Finally, some lower and upper bounds for the δ -numerical radius in the sectorial cases of coordinate operators are established, and these evaluations generalize some results known in the literature. Moreover, the obtained results offer a range of useful techniques that can facilitate further advancements in this field.

Mathematics subject classification (2020): 47A12, 47A30, 47A63, 15A60.

Keywords and phrases: δ -numerical radius, δ -Crawford number, direct sum of Hilbert space operators, sectorial operator.

REFERENCES

- [1] A. ABU-OMAR AND F. KITTANEH, *Numerical radius inequalities for $n \times n$ operator matrices*, Linear Algebra Appl. **468** (2015), 18–26.
- [2] O. AXELSSON, H. LU AND B. POLMAN, *On the numerical radius of matrices and its application to iterative solution methods*, Linear Multilinear Algebra **37** (1–3) (1994), 225–238.
- [3] Y. BEDRANI, F. KITTANEH AND M. SABABHEH, *Numerical radii of accretive matrices*, Linear Multilinear Algebra **69** (5) (2020), 957–970.
- [4] Y. M. BEREZANSKY, Z. G. SHEFTEL AND G. F. US, *Functional Analysis*, Birkhäuser Basel, Berlin, 1966.
- [5] P. BHUNIA AND K. PAUL, *Development of inequalities and characterization of equality conditions for the numerical radius*, Linear Algebra Appl. **630** (2021), 306–315.
- [6] F. F. BONSALL AND J. DUNCAN, *Numerical ranges II*, Cambridge University Press, London–New York, 1971.
- [7] N. P. DEKKER, *Joint numerical range and joint spectrum of Hilbert space operators*, Thesis, University of Amsterdam, 1969.
- [8] R. DRNOVŠEK AND V. MÜLLER, *On joint numerical radius II*, Linear Multilinear Algebra **62** (9) (2014), 1197–1204.
- [9] N. DUNFORD AND J. T. SCHWARTZ, *Linear Operators I*, Interscience Publishers, New York, 1958.
- [10] P. R. HALMOS, *A Hilbert Space Problem Book*, Springer-Verlag, New York–Berlin, 1982.
- [11] O. HIRZALLAH, F. KITTANEH AND K. SHEBRAWI, *Numerical radius inequalities for certain 2×2 operator matrices*, Integral Equations Operator Theory **71** (1) (2011), 129–147.
- [12] O. HIRZALLAH, F. KITTANEH AND K. SHEBRAWI, *Numerical radius inequalities for commutators of Hilbert space operators*, Numer. Funct. Anal. Optim. **32** (7) (2011), 739–749.
- [13] R. A. HORN AND C. R. JOHNSON, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.
- [14] Z. I. ISMAILOV AND R. Ö. MERT, *Gaps between some spectral characteristics of direct sum of Hilbert space operators*, Oper. Matrices **16** (2) (2022), 337–347.

- [15] F. KITTANEH, *A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix*, *Studia Math.* **158** (1) (2003), 11–17.
- [16] F. KITTANEH, *Numerical radius inequalities for Hilbert space operators*, *Studia Math.* **168** (1) (2005), 73–80.
- [17] H. MORADI AND M. SABABHEH, *New estimates for the numerical radius*, *Filomat* **35** (14) (2021), 4957–4962.
- [18] M. A. NAĬMARK AND S. V. FOMIN, *Continuous direct sums of Hilbert spaces and some of their applications*, *Uspehi Mat. Nauk* **10** 2 (64) (1955), 111–142.
- [19] E. OTKUN ÇEVİK, *Some numerical characteristics of direct sum of operators*, *Commun. Fac. Sci. Univ. Ank. Ser. A1. Math. Stat.* **69** (2) (2020), 1221–1227.
- [20] A. PATRA AND F. ROY, *On the estimation of q -numerical radius of Hilbert space operators*, *Oper. Matrices* **18** (2) (2024), 343–359.
- [21] M. SABABHEH AND H. MORADI, *More accurate numerical radius inequalities (I)*, *Linear Multilinear Algebra* **69** (10) (2019), 1964–1973.
- [22] M. SABABHEH AND H. MORADI, *More accurate numerical radius inequalities (II)*, *Linear Multilinear Algebra* **69** (5) (2019), 921–933.
- [23] S. A. SAMMOUR, F. KITTANEH AND M. SABABHEH, *A geometric approach to numerical radius inequalities*, *Linear Algebra Appl.* **652** (2022), 1–17.
- [24] J. G. STAMPFLI, *The norm of a derivation*, *Pacific J. Math.* **33** (1970), 737–747.
- [25] H. STANKOVIĆ, M. KRSTIĆ AND I. DAMNJANOVIĆ, *Some properties of the q -numerical radius*, *Linear Multilinear Algebra* **73** (8) (2025), 1736–1757.
- [26] P. Y. WU AND H. L. GAU, *Numerical ranges of Hilbert space operators*, *Encyclopedia Math. Appl.* **179**, Cambridge University Press, Cambridge, 2021.
- [27] T. YAMAZAKI, *On upper and lower bounds of the numerical radius and equality condition*, *Studia Math.* **178** (2007), 83–89.