

NEW INEQUALITIES FOR (\mathcal{M}_Ψ, h) -CONVEX
FUNCTIONS AND APPLICATIONS

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Abstract. Convexity plays a central role in analysis and inequality theory, and several generalizations such as h -convexity and (p, h) -convexity have been introduced to capture more flexible functional behaviors. In this paper, we further extend these ideas by investigating the notion of (\mathcal{M}_Ψ, h) -convexity, which unifies various known convexity classes through a mean transformation \mathcal{M}_Ψ and a control function h . We establish new refinements of Jensen-type inequalities for positive h -convex functions, including both direct and reversed versions. These results provide sharper bounds than the classical h -convexity inequality and naturally extend recent work of Ighachane *et al.* to a broader framework. In addition, we present new inequalities for τ -measurable operators, extending the theory to the setting of operator algebras.

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