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AN EXTENSION OF UCHIYAMA'S RESULT ASSOCIATED WITH AN ORDER PRESERVING OPERATOR INEQUALITY

TAKAYUKI FURUTA AND EIZABURO KAMEI

Abstract. Let A, B and C be positive invertible operators and also let r, s and t be non-negative real numbers such that $t \geqslant s$ and $(r,t) \neq (0,0)$. Then the following (I) and (II) hold and follows from each other.

(I) If $A^t \ll B^t \nabla_\lambda C^t$ (i.e., $\log A^t \leqslant \log(B^t \nabla_\lambda C^t\ ))$ for all $t \geqslant 0$, then

$$f(t) = \{A^{\frac{r}{2}}(B^t \nabla_\lambda C^t)A^{\frac{r}{2}}\}^{\frac{s+r}{t+r}} \quad \text{is an increasing function of} \quad t.$$

(II) If $A^t \gg B^t !_\lambda C^t$ (i.e., $\log A^t \geqslant \log(B^t !_\lambda C^t\ ))$ for all $t \geqslant 0$, then

$$h(t) = \{A^{\frac{r}{2}}(B^t !_\lambda C^t)A^{\frac{r}{2}}\}^{\frac{s+r}{t+r}} \quad \text{is a decreasing function of} \quad t,$$

where $B\nabla_\lambda C$ and $B!_\lambda C$ are the arithmetic mean and the harmonic mean respectively.
In particular we have

(I') *If* $A^t \ll B^t \nabla_\lambda C^t$, then

$$A^{\frac{r}{2}}(B^s \nabla_\lambda C^s)A^{\frac{r}{2}} \leqslant \{A^{\frac{r}{2}}(B^t \nabla_\lambda C^t)A^{\frac{r}{2}}\}^{\frac{s+r}{t+r}}$$

(II') *If* $A^t \gg B^t !_\lambda C^t$, then

$$A^{\frac{r}{2}}(B^s !_\lambda C^s)A^{\frac{r}{2}} \geqslant \{A^{\frac{r}{2}}(B^t !_\lambda C^t)A^{\frac{r}{2}}\}^{\frac{s+r}{t+r}}.$$

These are extensions of the recent results in Uchiyama [12].

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