

MONOTONICITY OF SEQUENCES INVOLVING GEOMETRIC MEANS OF POSITIVE SEQUENCES WITH MONOTONICITY AND LOGARITHMICAL CONVEXITY

BAI-NI GUO AND FENG QI

Abstract. Let f be a positive function such that $x[f(x+1)/f(x) - 1]$ is increasing on $[1, \infty)$, then the sequence $\left\{ \sqrt[n]{\prod_{i=1}^n f(i)} / f(n+1) \right\}_{n=1}^{\infty}$ is decreasing. If f is a logarithmically concave and positive function defined on $[1, \infty)$, then the sequence $\left\{ \sqrt[n]{\prod_{i=1}^n f(i)} / \sqrt{f(n)} \right\}_{n=1}^{\infty}$ is increasing. As consequences of these monotonicities, the lower and upper bounds for the ratio $\sqrt[n]{\prod_{i=k+1}^{n+k} f(i)} / \sqrt[n+m]{\prod_{i=k+1}^{n+k+m} f(i)}$ are obtained, where k is a nonnegative integer and m a natural number. Some applications are given.

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