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ON THE REAL LINEAR POLARIZATION CONSTANT PROBLEM

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Abstract. The present paper deals with lower bounds for the norm of products of linear forms. It has been proved by J. Arias-de-Reyna [2], that the so-called n^{th} linear polarization constant $c_n(\mathbb{C}^n)$ is $n^{n/2}$, for arbitrary $n \in \mathbb{N}$. The same value for $c_n(\mathbb{R}^n)$ is only conjectured. In a recent work A. Pappas and S. Révész prove that $c_n(\mathbb{R}^n) = n^{n/2}$ for $n \leqslant 5$. Moreover, they show that if the linear forms are given as $f_j(x) = \langle x, a_j \rangle$, for some unit vectors a_j $(1 \leqslant j \leqslant n)$, then the product of the f_j 's attains at least the value $n^{-n/2}$ at the normalized signed sum of the vectors $\{a_j\}_{j=1}^n$ having maximal length. Thus they asked whether this phenomenon remains true for arbitrary $n \in \mathbb{N}$. We show that for vector systems $\{a_j\}_{j=1}^n$ close to an orthonormal system, the Pappas-Révész estimate does hold true. Furthermore, among these vector systems the only system giving $n^{-n/2}$ as the norm of the product is the orthonormal system. On the other hand, for arbitrary vector systems we answer the question of A. Pappas and S. Révész in the negative when $n \in \mathbb{N}$ is large enough. We also discuss various further examples and counterexamples that may be instructive for further research towards the determination of $c_n(\mathbb{R}^n)$.

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