

## FUNDAMENTALS OF EQUILIBRIUM PROBLEMS

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*Abstract.* Equilibrium problems theory provides us with a unified, natural, innovative and general framework for studying a wide class of linear and nonlinear problems arising in finance, economics, image reconstructions, medical imaging, ecology, network analysis, transportation, elasticity, operations research and optimization. In this work, we consider some new classes of equilibrium problems in the framework of convexity, invexity,  $g$ -convexity and prox-regular convexity. We also study a class of equilibrium problems involving the nondifferentiability Lipschitz continuous functions, which is known as the hemiequilibrium problems. The auxiliary principle technique is used to suggest and analyze several iterative schemes for solving these classes of equilibrium problems. We consider the convergence analysis of these iterative algorithms under some mild conditions. We also introduce the concept of well-posedness for the equilibrium problems and obtain some interesting results. As special cases, we obtain several known and new results for variational inequalities and related optimization problems. Results obtained in this paper can be viewed as a nice and novel applications of the auxiliary principle technique in this fast growing and fascinating field.

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