

MONOTONICITY RESULTS FOR THE POLYGAMMA FUNCTIONS

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Abstract. In this paper, the monotonic results of the functions $x^c |\psi^{(n)}(x + \beta)|$ and $x\psi^{(n+1)}(x + \beta) / \psi^{(n)}(x + \beta)$ are established. Several by-products are obtained. Moreover, we prove that the function $\frac{d\psi(\ln x)}{dx}$ is completely monotonic.

Mathematics subject classification (2000): 33B15, 26A48.

Key words and phrases: digamma function, polygamma functions, Laplace transforms, completely monotonic.

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