

OPERATOR INEQUALITIES ON HILBERT C^* -MODULES VIA THE CAUCHY–SCHWARZ INEQUALITY

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Abstract. By means of a new Cauchy-Schwarz inequality in the framework of a semi-inner product C^* -module over a unital C^* -algebra, we discuss some operator inequalities on a Hilbert C^* -module, for example, Kantorovich inequality, Pólya-Szegő inequality, the covariance-variance inequality, Ozeki-Izumino-Mori-SEO inequality, Wielandt inequality, Hienz-Kato-Furuta inequality and Malamud inequality.

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REFERENCES

- [1] T. ANDO, *Topics on Operator Inequalities*, Lecture notes (mimeographed), Hokkaido Univ., Sapporo, 1978.
- [2] L. ARAMBASIĆ, D. BAKIĆ AND M. S. MOSLEHIAN, *A treatment of the Cauchy–Schwarz inequality in C^* -modules*, *J. Math. Anal. Appl.*, **381** (2011) 546–556.
- [3] C. W. CHIANG, *On a generalization of the Wielandt inequality*, *Soochow J. Math.*, **21**, No. 1, (1995), 117–120.
- [4] S. S. DRAGOMIR, *A counterpart of Schwarz inequality in inner product spaces*, *RGMIA Res. Rep. Coll.*, **6** (2003), Suppl. Article 18.
- [5] S. S. DRAGOMIR, *New inequalities of the Kantorovich type for bounded linear operators in Hilbert spaces*, *Linear Algebra Appl.*, **428** (2008), 2750–2760.
- [6] J. I. FUJII, *Operator-valued inner product and operator inequalities*, *Banach J. Math. Anal.*, **2** (2008), no. 2, 59–67.
- [7] J. I. FUJII, M. FUJII, M. S. MOSLEHIAN, J. PEČARIĆ AND Y. SEO, *Reverse Cauchy–Schwarz type inequalities in pre-inner product C^* -modules*, *Hokkaido Math. J.*, **40** (2011), 1–17.
- [8] J. I. FUJII, M. FUJII, M. S. MOSLEHIAN AND Y. SEO, *Cauchy–Schwarz inequality in semi-inner product C^* -modules via polar decomposition*, *J. Math. Anal. Appl.*, **394** (2012), 835–840.
- [9] J. I. FUJII, R. NAKAMOTO AND Y. SEO, *Operator inequalities of Malamud and Wielandt*, *Linear Algebra Appl.*, **326** (2001), 101–109.
- [10] M. FUJII, T. FURUTA, R. NAKAMOTO AND S.-E. TAKAHASI, *Operator inequalities and covariance in noncommutative probability*, *Math. Japon.*, **46** (1997), 317–320.
- [11] M. FUJII, S. IZUMINO, R. NAKAMOTO AND Y. SEO, *Operator inequalities related to Cauchy-Schwarz and Hölder-McCarthy inequalities*, *Nihonkai Math. J.*, **8** (1997), 117–122.
- [12] M. FUJII, Y. KATAYAMA AND R. NAKAMOTO, *Generalization of the Wielandt theorem*, *Math. Japon.*, **49** (1999), 217–222.
- [13] M. FUJII AND Y. SEO, *Wielandt type extensions of the Heinz-Kato-Furuta inequality*, *Operator Theory: Adv. Appl.*, **127** (2001), 267–277.
- [14] T. FURUTA, *An extension of the Heinz-Kato theorem*, *Proc. Amer. Math. Soc.*, **120** (1994), 785–787.
- [15] T. FURUTA, *Invitation to Linear Operators*, Taylor & Francis, London, 2001.
- [16] T. FURUTA, J. MIČIĆ HOT, J. PEČARIĆ AND Y. SEO, *Mond-Pečarić Method in Operator Inequalities*, *Monographs in Inequalities 1*, Element, Zagreb, 2005.

- [17] W. GREUB AND W. RHEINBOLDT, *On a generalization of an inequality of L. V. Kantorovich*, Proc. Amer. Math. Soc., **10** (1959), 407–415.
- [18] M. HAMANA, *Partial *-automorphisms, normalizers, and submodules in monotone complete C^* -algebras*, Canad. J. Math., **58** (2006), 1144–1202.
- [19] R. A. HORN AND C. R. JOHNSON, *Matrix Analysis*, Cambridge Univ. Press, 1985.
- [20] D. ILIŠEVIĆ AND S. VAROŠANEC, *On the Cauchy-Schwarz inequality and its reverse in semi-inner product C^* -modules*, Banach J. Math. Anal., **1** (2007), no. 1, 78–84.
- [21] S. IZUMINO, H. MORI AND Y. SEO, *On Ozeki's inequality*, J. Inequal. Appl., **2** (1998), 235–253.
- [22] L. V. KANTOROVICH, *Functional analysis and applied mathematics*, Uspehi Mat. Nauk., **3** (1948), pp. 89–185. Translated from the Russian by Curtis D. Benster, National Bureau of Standards, Report 1509, March 7, 1952.
- [23] I. KAPLANSKY, *Modules over operator algebras*, Amer. J. Math., **75** (1953), 839–858.
- [24] E. C. LANCE, *Hilbert C^* -Modules*, London Math. Soc., Lecture Note Series 210, Cambridge Univ. Press, 1995.
- [25] S. M. MALAMUD, *On a class of multiparameter perturbations of positive definite operators with fixed bounded on their spectrum*, Linear Algebra Appl., **257** (1998), 239–257.
- [26] S. M. MOSLEHIAN, R. NAKAMOTO AND Y. SEO, *A Diaz-Metcalf type inequality for positive linear maps and its applications*, Electron J. Linear Algebra, **22** (2011), 179–190.
- [27] M. S. MOSLEHIAN AND L.-E. PERSSON, *Reverse Cauchy–Schwarz inequalities for positive C^* -valued sesquilinear forms*, Math. Inequal. Appl., **4** (2009), no. 12, 701–709.
- [28] M. NAGISA, *A topic in almost commuting matrix (in Japanese)*, RIMS, Kokyuroku 1046, Research Institute for Mathematical Sciences Kyoto University, (1998), 15–28.
- [29] C. P. NICULESCU, *Converses of the Cauchy–Schwarz inequality in the C^* -framework*, An. Univ. Craiova, Ser. Mat. Inf., **26** (1999), 22–28.
- [30] N. OZEKI, *On the estimation of the inequalities by the maximum, or minimum values (in Japanese)*, J. College Arts Sci. Chiba Univ., **5** (1968), 199–203.
- [31] G. PÓLYA AND G. SZEGÖ, *Aufgaben und Lehrsätze der Analysis*, **1**, Berlin, 1925.
- [32] Y. SEO, *Variance in noncommutative probability*, Math. Japon., **46** (1997), 439–444.