

ON JORDAN'S, REDHEFFER'S AND WILKER'S INEQUALITY

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Abstract. In this paper, the authors offer new Jordan, Redheffer and Wilker type inequalities, along with refinements and converses. Connections with Euler's gamma function are pointed out, too.

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