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ON A FUNCTIONAL EQUATION RELATED TO TWO–VARIABLE CAUCHY MEANS

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Abstract. In this paper, we are dealing with the solution of the functional equation

$$\varphi\left(\frac{x+y}{2}\right)(f(x)-f(y)) = F(x)-F(y),$$

concerning the unknown functions φ, f and F defined on a same open subinterval of the reals. Improving the previous results related to this topic, we describe the solution triplets (φ, f, F) assuming only the continuity of φ.

As an application, under natural conditions, we also solve the equality problem of two-variable Cauchy means and two-variable quasi-arithmetic means.

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