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ON THE VARIATION OF THE DISCRETE MAXIMAL OPERATOR

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Abstract. In this note we study the endpoint regularity properties of the discrete nontangential fractional maximal operator

$$M_{\alpha,\beta}f(n) = \sup_{\substack{r\in\mathbb{N}\\ |m-n|\leqslant\beta r}} \frac{1}{(2r+1)^{1-\alpha}}\sum_{k=-r}^{r}|f(m+k)|,$$

where $\alpha\in[0,1)$, $\beta\in[0,\infty)$ and $\mathbb{N}=\{0,1,2,\dots,\}$, covering the discrete centered Hardy-Littlewood maximal operator and its fractional variant. More precisely, we establish the sharp boundedness and continuity for $M_{\alpha,\beta}$ from $\ell^1(\mathbb{Z})$ to $\mathrm{BV}(\mathbb{Z})$. When $\alpha=0$, we prove that the operator $M_{\alpha,\beta}$ is bounded and continuous on $\mathrm{BV}(\mathbb{Z})$. Here $\mathrm{BV}(\mathbb{Z})$ denotes the set of functions of bounded variation defined on $\mathbb{Z}$. Our main results represent generalizations as well as natural extensions of many previously known ones.

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