

ON WEIGHTED HARDY-TYPE INEQUALITIES

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Abstract. We revisit weighted Hardy-type inequalities employing an elementary ad hoc approach that yields explicit constants. We also discuss the infinite sequence of power weighted Birman–Hardy–Rellich-type inequalities and derive an operator-valued version thereof.

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