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ON EXTREMALS FOR THE TRUDINGER–MOSER INEQUALITY WITH VANISHING WEIGHT IN THE N–DIMENSIONAL UNIT BALL

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Abstract. In this paper, we study the extremal function for the Trudinger-Moser inequality with vanishing weight in the unit ball $\mathbb{B} \subset \mathbb{R}^N$ $(N \geqslant 3)$. To be exact, let $\mathscr{S}$ be the set of all decreasing radially symmetrical functions and $\alpha_N = N\omega_{N-1}^{1/(N-1)}$, where ω_{N-1} is the area of the unit sphere in $\mathbb{R}^N$. Suppose h is a nonnegative radially symmetrical function belonging to $C^0(\overline{\mathbb{B}})$ satisfying $h(x) > 0$ in $\overline{\mathbb{B}} \setminus \{0\}$ and $h(x)|x|^{-N\beta} \to 1$ as $x \to 0$ for some real number $\beta \geqslant 0$. By means of blow-up analysis, we prove that the supremum

$$\Lambda_\beta := \sup_{u \in W_0^{1,N}(\mathbb{B}) \cap \mathscr{S},\, \|\nabla u\|_N \leqslant 1} \int_{\mathbb{B}} \exp\left\{\alpha_N (1+\beta)\, |u|^{\frac{N}{N-1}}\right\} h(x)\, dx$$

can be attained by some $u_0 \in W_0^{1,N}(\mathbb{B}) \cap \mathscr{S}$ with $\|\nabla u_0\|_N = 1$. This improves a recent result of Yang-Zhu [39].

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