

ASYMPTOTIC EXPANSION AND BOUNDS FOR COMPLETE ELLIPTIC INTEGRALS

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Abstract. In the article, we present several new bounds for the the complete elliptic integrals $\mathcal{K}(r) = \int_0^{\pi/2} (1-r^2 \sin^2 \theta)^{-1/2} d\theta$ and $\mathcal{E}(r) = \int_0^{\pi/2} (1-r^2 \sin^2 \theta)^{1/2} d\theta$, and find an asymptotic expansion for $\mathcal{K}(r)$ as $r \rightarrow 1$, which are the refinements and improvements of the previously well-known results.

Mathematics subject classification (2010): 33E05, 26E60.

Keywords and phrases: Asymptotic expansion, complete elliptic integrals, arithmetic-geometric mean, infinite series, bound.

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