

ON HARDY TYPE INEQUALITIES FOR WEIGHTED QUASIDEVIATION MEANS

ZSOLT PÁLES AND PAWEŁ PASTECZKA

Abstract. Using recent results concerning the homogenization and the Hardy property of weighted means, we establish sharp Hardy constants for concave and monotone weighted quasideviation means and for a few particular subclasses of this broad family. More precisely, for a mean \mathcal{D} like above and a sequence (λ_n) of positive weights such that $\lambda_n/(\lambda_1 + \dots + \lambda_n)$ is nondecreasing, we determine the smallest number $H \in (1, +\infty]$ such that

$$\sum_{n=1}^{\infty} \lambda_n \mathcal{D}((x_1, \dots, x_n), (\lambda_1, \dots, \lambda_n)) \leq H \cdot \sum_{n=1}^{\infty} \lambda_n x_n \text{ for all } x \in \ell_1(\lambda).$$

It turns out that H depends only on the limit of the sequence $(\lambda_n/(\lambda_1 + \dots + \lambda_n))$ and the behaviour of the mean \mathcal{D} near zero.

Mathematics subject classification (2010): 26D15, 26E60, 39B62, 40D25.

Keywords and phrases: Weighted mean, Hardy inequality, Hardy constant, quasiarithmetic mean, quasideviation mean, Jensen concavity.

REFERENCES

- [1] M. BAJRAKTAREVIĆ, *Sur une équation fonctionnelle aux valeurs moyennes*, Glasnik Mat.-Fiz. Astronom. Društvo Mat. Fiz. Hrvatske Ser. II, 13:243–248, 1958.
- [2] F. BERNSTEIN AND G. DOETSCH, *Zur Theorie der konvexen Funktionen*, Math. Ann., 76(4):514–526, 1915.
- [3] T. CARLEMAN, *Sur les fonctions quasi-analitiques*, Conférences faites au cinquième congrès des mathématiciens scandinaves, Helsinki, page 181–196, 1932.
- [4] E. T. COPSON, *Note on Series of Positive Terms*, J. London Math. Soc., s1-2(1):9–12, 1927.
- [5] Z. DARÓCZY, *A general inequality for means*, Aequationes Math., 7(1):16–21, 1971.
- [6] Z. DARÓCZY, *Über eine Klasse von Mittelwerten*, Publ. Math. Debrecen, 19:211–217 (1973), 1972.
- [7] Z. DARÓCZY AND L. LOSONCZI, *Über den Vergleich von Mittelwerten*, Publ. Math. Debrecen, 17:289–297 (1971), 1970.
- [8] Z. DARÓCZY AND ZS. PÁLES, *On comparison of mean values*, Publ. Math. Debrecen, 29(1-2):107–115, 1982.
- [9] Z. DARÓCZY AND ZS. PÁLES, *Multiplicative mean values and entropies*, Functions, series, operators, Vol. I, II (Budapest, 1980), page 343–359. North-Holland, Amsterdam, 1983.
- [10] J. DUNCAN AND C. M. MCGREGOR, *Carleman's Inequality*, Amer. Math. Monthly, 110(5):424–431, 2003.
- [11] E. B. ELLIOTT, *A simple exposition of some recently proved facts as to convergency*, J. London Math. Soc., 1:93–96, 1926.
- [12] C. GINI, *Di una formula compressiva delle medie*, Metron, 13:3–22, 1938.
- [13] G. H. HARDY, *Note on a theorem of Hilbert*, Math. Z., 6:314–317, 1920.
- [14] G. H. HARDY, J. E. LITTLEWOOD, AND G. PÓLYA, *Inequalities*, Cambridge University Press, Cambridge, 1934. (first edition), 1952 (second edition).
- [15] K. KNOPP, *Über Reihen mit positiven Gliedern*, J. London Math. Soc., 3:205–211, 1928.
- [16] A. KUFNER, L. MALIGRANDA, AND L.-E. PERSSON, *The Hardy Inequality: About Its History and Some Related Results*, Vydavatelský servis, 2007.

- [17] E. LANDAU, *A note on a theorem concerning series of positive terms*, J. London Math. Soc., 1:38–39, 1921.
- [18] L. LOSONCZI, *Über den Vergleich von Mittelwerten die mit Gewichtsfunktionen gebildet sind*, Publ. Math. Debrecen, 17:203–208 (1971), 1970.
- [19] L. LOSONCZI, *Subadditive Mittelwerte*, Arch. Math. (Basel), 22:168–174, 1971.
- [20] L. LOSONCZI, *Subhomogene Mittelwerte*, Acta Math. Acad. Sci. Hungar., 22:187–195, 1971.
- [21] L. LOSONCZI, *Über eine neue Klasse von Mittelwerten*, Acta Sci. Math. (Szeged), 32:71–81, 1971.
- [22] L. LOSONCZI, *General inequalities for nonsymmetric means*, Aequationes Math., 9:221–235, 1973.
- [23] L. LOSONCZI, *Inequalities for integral mean values*, J. Math. Anal. Appl., 61(3):586–606, 1977.
- [24] P. MULHOLLAND, *On the generalization of Hardy's inequality*, J. London Math. Soc., 7:208–214, 1932.
- [25] P. PASTECZKA, *When is a family of generalized means a scale?*, Real Anal. Exchange, 38(1):193–209, 2012/13.
- [26] J. E. PEČARIĆ AND K. B. STOLARSKY, *Carleman's inequality: history and new generalizations*, Aequationes Math., 61(1–2):49–62, 2001.
- [27] Zs. PÁLES, *Characterization of quasideviation means*, Acta Math. Acad. Sci. Hungar., 40(3–4):243–260, 1982.
- [28] Zs. PÁLES, *On complementary inequalities*, Publ. Math. Debrecen, 30(1–2):75–88, 1983.
- [29] Zs. PÁLES, *Inequalities for comparison of means*, In W. Walter, editor, General Inequalities, 4 (Oberwolfach, 1983), volume 71 of International Series of Numerical Mathematics, page 59–73. Birkhäuser, Basel, 1984.
- [30] Zs. PÁLES, *Ingham Jessen's inequality for deviation means*, Acta Sci. Math. (Szeged), 49(1–4):131–142, 1985.
- [31] Zs. PÁLES, *General inequalities for quasideviation means*, Aequationes Math., 36(1):32–56, 1988.
- [32] Zs. PÁLES, *On a Pexider-type functional equation for quasideviation means*, Acta Math. Hungar., 51(1–2):205–224, 1988.
- [33] Zs. PÁLES, *On homogeneous quasideviation means*, Aequationes Math., 36(2–3):132–152, 1988.
- [34] Zs. PÁLES, *A Hahn-Banach theorem for separation of semigroups and its applications*, Aequationes Math., 37(2–3):141–161, 1989.
- [35] Zs. PÁLES AND P. PASTECZKA, *On the homogenization of means*, Acta Math. Hungar. 159(2):537–562, 2019.
- [36] Zs. PÁLES AND P. PASTECZKA, *Characterization of the Hardy property of means and the best Hardy constants*, Math. Ineq. Appl., 19:1141–1158, 2016.
- [37] Zs. PÁLES AND P. PASTECZKA, *On Kedlaya type inequalities for weighted means*, J. Inequal. Appl., 2018(99), 2018.
- [38] Zs. PÁLES AND P. PASTECZKA, *On the best Hardy constant for quasi-arithmetic means and homogeneous deviation means*, Math. Inequal. Appl., 21:585–599, 2018.
- [39] Zs. PÁLES AND P. PASTECZKA, *On Hardy type inequalities for weighted means*, Banach J. Math. Anal., 13:217–233, 2019.