

POSITIVE DEFINITENESS ON PRODUCTS VIA GENERALIZED STIELTJES AND OTHER FUNCTIONS

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Abstract. Let (X, ρ) and (Y, σ) be quasi-metric spaces and λ a fixed positive real number. This paper establishes the positive definiteness of functions of the form

$$G_r(t, u) = \frac{1}{h(u)^r} f\left(\frac{g(t)}{h(u)}\right), \quad (t, u) \in \{\rho(x, x') : x, x' \in X\} \times \{\sigma(y, y') : y, y' \in Y\},$$

on $X \times Y$, where $r \geq \lambda$, f belongs to the convex cone of all generalized Stieltjes functions of order λ , and g and h are positive valued conditionally negative definite functions on (X, ρ) and (Y, σ) , respectively. As a bypass, it establishes the positive definiteness of functions of the form

$$H_r(t, u) = \frac{1}{g(t)^r} f\left(\frac{g(t)}{h(u)}\right), \quad (t, u) \in \{\rho(x, x') : x, x' \in X\} \times \{\sigma(y, y') : y, y' \in Y\},$$

for a generalized complete Bernstein function f of order λ , under the same assumptions on r , g and h . The paper also provides necessary and sufficient conditions for the strict positive definiteness of the two models when the spaces involved are metric. The two results yield additional methods to construct positive definite and strictly positive definite functions on a product of metric spaces by integral transforms.

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