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EXACT CONVERSES TO A REVERSE AM—GM INEQUALITY, WITH APPLICATIONS TO SUMS OF INDEPENDENT RANDOM VARIABLES AND (SUPER)MARTINGALES

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Abstract. For every given real value of the ratio $\mu := A_X/G_X > 1$ of the arithmetic and geometric means of a positive random variable X and every real $v > 0$, exact upper bounds on the right- and left-tail probabilities $\mathsf{P}(X/G_X \geqslant v)$ and $\mathsf{P}(X/G_X \leqslant v)$ are obtained, in terms of μ and v. In particular, these bounds imply that $X/G_X \to 1$ in probability as $A_X/G_X \downarrow 1$. Such a result may be viewed as a converse to a reverse Jensen inequality for the strictly concave function $f = \ln$, whereas the well-known Cantelli and Chebyshev inequalities may be viewed as converses to a reverse Jensen inequality for the strictly concave quadratic function $f(x) \equiv -x^2$. As applications of the mentioned new results, improvements of the Markov, Bernstein–Chernoff, sub-Gaussian, and Bennett–Hoeffding probability inequalities are given.

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