

## ON WEIGHTED HARDY INEQUALITY WITH TWO-DIMENSIONAL RECTANGULAR OPERATOR — EXTENSION OF THE E. SAWYER THEOREM

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*Abstract.* A characterization is obtained for those pairs of weights  $v$  and  $w$  on  $\mathbb{R}_+^2$ , for which the two-dimensional rectangular integration operator is bounded from a weighted Lebesgue space  $L_v^p(\mathbb{R}_+^2)$  to  $L_w^q(\mathbb{R}_+^2)$  for  $1 < p < q < \infty$ , which is an essential complement to E. Sawyer's result [13] given for  $1 < p \leq q < \infty$ . Besides, we demonstrate that the E. Sawyer theorem is actual if  $p = q$  only, for  $p < q$  the criterion is the finiteness of the Muckenhoupt-type constant. The case  $q < p$  is also discussed.

*Mathematics subject classification (2020):* 26D10, 47G10.

*Keywords and phrases:* Rectangular integration operator, Hardy inequality, weighted Lebesgue space.

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