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A TRUDINGER–MOSER INEQUALITY WITH MEAN VALUE ZERO ON A COMPACT RIEMANN SURFACE WITH BOUNDARY

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Abstract. In this paper, on a compact Riemann surface (Σ, g) with smooth boundary $\partial\Sigma$, we concern a Trudinger-Moser inequality with mean value zero. To be exact, let $\lambda_1(\Sigma)$ denotes the first eigenvalue of the Laplace-Beltrami operator with respect to the zero mean value condition and $\mathscr{S} = \left\{u \in W^{1,2}(\Sigma, g) : \|\nabla_g u\|_2^2 \leqslant 1 \text{ and } \int_\Sigma u \mathrm{d}v_g = 0\right\}$, where $W^{1,2}(\Sigma, g)$ is the usual Sobolev space, $\|\cdot\|_2$ denotes the standard L^2-norm and ∇_g represent the gradient. By the method of blow-up analysis, we obtain

$$\sup_{u\in\mathscr{S}} \int_\Sigma e^{2\pi u^2\left(1+\alpha\|u\|_2^2\right)} \mathrm{d}v_g < +\infty, \ \forall\, 0 \leqslant \alpha < \lambda_1(\Sigma);$$

when $\alpha \geqslant \lambda_1(\Sigma)$, the supremum is infinite. Moreover, we prove the supremum is attained by a function $u_\alpha \in C^\infty\left(\overline{\Sigma}\right) \cap \mathscr{S}$ for sufficiently small $\alpha > 0$. Based on the similar work in the Euclidean space, which was accomplished by Lu-Yang [19], we strengthen the result of Yang [29].

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