

## MODULUS OF CONTINUITY OF THE QUANTUM $f$ -ENTROPY WITH RESPECT TO THE TRACE DISTANCE

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*Abstract.* A well-known result due to Fannes is a certain upper bound on the modulus of continuity of the von Neumann entropy with respect to the trace distance between density matrices; this distance is the maximum probability of distinguishing between the corresponding quantum states. Much more recently, Audenaert obtained an exact expression of this modulus of continuity.

In the present note, Audenaert's result is extended to a broad class of entropy functions indexed by arbitrary continuous convex functions  $f$  in place of the Shannon–von Neumann function  $x \mapsto x \log_2 x$ . The proof is based on the Schur majorization.

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