

## ON THE NUMBER OF REAL ZEROS OF REAL ENTIRE FUNCTIONS WITH A NON-DECREASING SEQUENCE OF THE SECOND QUOTIENTS OF TAYLOR COEFFICIENTS

THU HIEN NGUYEN\* AND ANNA VISHNYAKOVA

*Abstract.* For an entire function  $f(z) = \sum_{k=0}^{\infty} a_k z^k$ ,  $a_k > 0$ , we define the sequence of the second quotients of Taylor coefficients  $Q := \left( \frac{a_k^2}{a_{k-1}a_{k+1}} \right)_{k=1}^{\infty}$ . We find new necessary conditions for a function with a non-decreasing sequence  $Q$  to belong to the Laguerre-Pólya class of type I. We also estimate the possible number of non-real zeros for a function with a non-decreasing sequence  $Q$ .

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