

REGULARITY OF COMMUTATORS OF MULTILINEAR MAXIMAL OPERATORS WITH LIPSCHITZ SYMBOLS

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Abstract. We study the regularity properties for commutators of multilinear fractional maximal operators. More precisely, let $m \geq 1$, $0 \leq \alpha < mn$ and $\vec{b} = (b_1, \dots, b_m)$ with each b_i belonging to the Lipschitz space $\text{Lip}(\mathbb{R})$, we denote by $[\vec{b}, \mathfrak{M}_\alpha]$ (resp., $\mathfrak{M}_{\alpha, \vec{b}}$) the commutator of the multilinear fractional maximal operator \mathfrak{M}_α with \vec{b} (resp., the multilinear fractional maximal commutators). When $\alpha = 0$, we denote $[\vec{b}, \mathfrak{M}_\alpha] = [\vec{b}, \mathfrak{M}]$ and $\mathfrak{M}_{\alpha, \vec{b}} = \mathfrak{M}_{\vec{b}}$. We show that for $0 < s < 1$, $1 < p_1, \dots, p_m, p, q < \infty$, $1/p = 1/p_1 + \dots + 1/p_m$, both $[\vec{b}, \mathfrak{M}]$ and $\mathfrak{M}_{\vec{b}}$ are bounded and continuous from $W^{s, p_1}(\mathbb{R}^n) \times \dots \times W^{s, p_m}(\mathbb{R}^n)$ to $W^{s, p}(\mathbb{R}^n)$, from $F_s^{p_1, q}(\mathbb{R}^n) \times \dots \times F_s^{p_m, q}(\mathbb{R}^n)$ to $F_s^{p, q}(\mathbb{R}^n)$ and from $B_s^{p_1, q}(\mathbb{R}^n) \times \dots \times B_s^{p_m, q}(\mathbb{R}^n)$ to $B_s^{p, q}(\mathbb{R}^n)$. It was also shown that for $0 \leq \alpha < mn$, $1 < p_1, \dots, p_m, q < \infty$ and $1/q = 1/p_1 + \dots + 1/p_m - \alpha/n$, both $[\vec{b}, \mathfrak{M}]$ and $\mathfrak{M}_{\vec{b}}$ are bounded from $W^{1, p_1}(\mathbb{R}^n) \times \dots \times W^{1, p_m}(\mathbb{R}^n)$ to $W^{1, q}(\mathbb{R}^n)$.

Mathematics subject classification (2020): 42B25, 46E35.

Keywords and phrases: Multilinear maximal operator, commutator, boundedness, Sobolev space, Triebel-Lizorkin space, Besov space.

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