Mathematical
Inequalities
& Applications
Volume 25, Number 2 (2022), 335–358 doi:10.7153/mia-2022-25-20

A NOTE ON THE GENERALIZED HAUSDORFF AND PACKING MEASURES OF PRODUCT SETS IN METRIC SPACE

RIHAB GUEDRI* AND NAJMEDDINE ATTIA

Abstract. Let μ and ν be two Borel probability measures on two separable metric spaces $\mathbb{X}$ and $\mathbb{Y}$ respectively. For h,g be two Hausdorff functions and $q \in \mathbb{R}$, we introduce and investigate the generalized pseudo-packing measure $\mathscr{R}_{\mu}^{q,h}$ and the weighted generalized packing measure $\mathscr{Q}_{\mu}^{q,h}$ to give some product inequalities :

$$\mathscr{H}_{\mu\times\nu}^{q,hg}(E\times F) \leqslant \mathscr{H}_{\mu}^{q,h}(E)\,\mathscr{R}_{\nu}^{q,g}(F) \leqslant \mathscr{R}_{\mu\times\nu}^{q,hg}(E\times F)$$

and

$$\mathscr{P}_{\mu\times\nu}^{q,hg}(E\times F) \leqslant \mathscr{Q}_{\mu}^{q,h}(E)\,\mathscr{P}_{\nu}^{q,g}(F)$$

for all $E \subseteq \mathbb{X}$ and $F \subseteq \mathbb{Y}$, where $\mathscr{H}_{\mu}^{q,h}$ and $\mathscr{P}_{\mu}^{q,h}$ is the generalized Hausdorff and packing measures respectively. As an application, we prove that under appropriate geometric conditions, there exists a constant c such that

$$\mathscr{H}_{\mu\times\nu}^{q,hg}(E\times F) \leqslant c\,\mathscr{H}_{\mu}^{q,h}(E)\,\mathscr{P}_{\nu}^{q,g}(F)$$
$$\mathscr{H}_{\mu}^{q,h}(E)\,\mathscr{P}_{\nu}^{q,g}(F) \leqslant c\,\mathscr{P}_{\mu}^{q,hg}(E\times F)$$
$$\mathscr{P}_{\mu\times\nu}^{q,hg}(E\times F) \leqslant c\,\mathscr{P}_{\mu}^{q,h}(E)\,\mathscr{P}_{\nu}^{q,g}(F).$$

These appropriate inequalities are more refined than well know results since we do no assumptions on μ,ν,h and g.

Mathematics subject classification (2020): 28A78, 28A80.
Keywords and phrases: Generalized Hausdorff and packing measures, weighted measure, product sets.

REFERENCES

[1] N. ATTIA, H. JEBALI AND M. BEN HADJ KHLIFA, *A note on fractal measures of cartesian product sets*, Bulletin of the Malaysian Mathematical Sciences Society **44** (2021), 4383–4404.
[2] A. S. BESICOVITCH AND P. A. P. MOHAN, *The measure of product and cylinder sets*, J. Lond. Math. Soc. **20** (1945), 110–120.
[3] R. O. DAVIES, *Measures not approximable or not specifiable by means of balls. Mathematika*, Mathematika **18** (2) (1971), 157–160.
[4] G. A. EDGAR, *Packing Measure in General Metric Space*, Real Anal. Exchange 26 (2001), 831–852.
[5] G. A. EDGAR, *Integral, probability, and fractal measures*, Springer-Verlag, New York, (1998).
[6] G. A. EDGAR, *Centred densities and fractal measures*, New York J. Math. **13** (2007), 33–87.
[7] H. FEDERER, *Geometric Measure Theory*, Springer-Verlag (1969).
[8] D.-J. FENG, S. HUA AND Z.-Y. WEN, *The pointwise densities of the Cantor measure*, J. Math. Anal. Appl. **250** (2000), 692–705.
[9] S. JURINA, N. MACGREGOR, A. MITCHELL, L. OLSEN AND A. STYLIANOU, *On the Hausdorff and packing measures of typical compact metric spaces*, Aequationes Math. **92** (2018), 709–735.
[10] O. GUIZANI, A. MAHJOUB AND N. ATTIA *On the Hewitt-Stromberg measure of product sets*, Annali di Matematica Pura ed Applicata (1923) **200**, 2 (2020), 867–879.
[11] O. GUIZANI, A. MAHJOUB AND N. ATTIA *Some relations between Hewitt-Stromberg premeasure and Hewitt-Stromberg measure*, to appear, Filomat.

Paper MIA-25-20

[12] O. GUIZANI AND N. ATTIA *A note on scaling properties of Hewitt Stromberg measure*, to appear, Filomat.
[13] H. HAASE, *The packing theorem and packing measure*, Math. Nachr. **146** (1990), 77–84.
[14] H. HAASE, *Dimension of measures*, Ada Universitatis Carolinae – Mathematica et Physica **31** (1990), 29–34.
[15] H. HAASE, *On the dimension of product measures*, Mathematika **37** (1990), 316–323.
[16] J. D. HOWROYD, *On Hausdorff and packing dimension of product spaces*, Math. Proc. Cambridge Philos. Soc. **119** (1996) 715–727.
[17] J. D. HOWROYD, *On the theory of Hausdorff measure in metric spaces*, Ph. D. thesis, University College London, (1994).
[18] X. HU AND S. J. TAYLOR, *Fractal properties of products and projections of measures in R*, Math. Proc. Cambridge Philos. Soc. **115** (1994), 527–544.
[19] B. JIA, Z. ZHOU, Z. ZHU AND J. LUO, *The packing measure of the Cartesian product of the middle third Cantor set with itself*, J. Math. Anal. Appl. **288** (2003) 424–441.
[20] X. JIANG, Q. LIU AND Z. WEN, *An intermediate value property of fractal dimensions of Cartesian product*, Fractals **25** (06) (2017) 1750052.
[21] H. JOYCE AND D. PREISS, *On the existence of subsets of positive finite packing measure*, Mathematika 42 (1995) 14–24.
[22] J. D. KELLY, *A method for constructing measures appropriate for the study of Cartesian products*, Proc. Lond. Math. Soc. (3) **26** (1973), 521–546.
[23] J. M. MARSTRAND, *The dimension of Cartesian product sets*, Math. Proc. Cambridge Philos. Soc. **50** (1954), 198–202.
[24] P. MATTILA, *Geometry of sets and Measures in Euclidian Spaces: Fractals and Rectifiability*, Cambridge University Press (1995).
[25] D. G. LARMAN, *A new theory of dimension*, Proc. Lond. Math. Soc. **17** (1967), 178–192.
[26] Y. PESIN, *Dimension theory in dynamical systems, Contemporary views and applications*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, (1997).
[27] L. OLSEN, *A multifractal formalism*, Advances in Mathematics, **116** (1995), 82–196.
[28] L. OLSEN, *Multifractal dimensions of product measures*, Math. Proc. Cambridge Philos. Soc. **120** (1996), 709–734.
[29] X. S. RAYMOND AND C. TRICOT, *Packing regularity of sets in n-space*, Math. Proc. Cambridge Philos. Soc. **103** (1988), 133–145.
[30] B. S. THOMSON, *Construction of measures in metric spaces*, J. Lond. Math. Soc. **14** (1976), 21–24.
[31] C. TRICOT, *Two definitions of fractional dimension*, Math. Proc. Cambridge Philos. Soc. **91** (1982) 57–74.
[32] S. WEN AND Z. WEN, *Note on packing and weak-packing measures with Hausdorff functions*, J. Math. Anal. Appl. **320** (2006), 482–488.
[33] C. WEI, S. WEN AND Z. WEN, *Remark on dimension of cartesian product sets*, Fractals, **24** 3 (2016).
[34] S. WEN AND M. WU, *Relations between packing premeasure and measure on metric space*, Acta Mathematica Scientia. **27** (2007), 137–144.
[35] O. ZINDULKA, *Packing measures and dimensions on Cartesian products*, Publ. Mat. **57** (2013), 393–420.