

GENERALIZED GAGLIARDO–NIRENBERG INEQUALITIES VIA MURAMATU’S INTEGRAL FORMULA

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Abstract. We derive three generalized Gagliardo-Nirenberg inequalities in Lorentz, BMO and homogeneous Lipschitz spaces. They have the forms

$$\|\nabla^k f\|_{L^{p,\alpha}(\mathbb{R}^d)} \lesssim \|f\|_{L^{q,\infty}(\mathbb{R}^d)}^\theta \|\nabla^m f\|_{L^{r,\infty}(\mathbb{R}^d)}^{1-\theta},$$

$$\|\nabla^k f\|_{L^{p,\alpha_1}(\mathbb{R}^d)} \lesssim \|f\|_{L^{q,\alpha_2}(\mathbb{R}^d)}^\theta \|\nabla^m f\|_{BMO(\mathbb{R}^d)}^{1-\theta},$$

$$\|f\|_{L^{p,\alpha}(\mathbb{R}^d)} \lesssim \|f\|_{L^{q,\infty}(\mathbb{R}^d)}^\theta \|f\|_{\dot{\Lambda}_\eta(\mathbb{R}^d)}^{1-\theta},$$

whose parameters satisfy specific conditions. We use the so-called Muramatu’s integral formula as the main approach throughout the paper.

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