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SHARP ESTIMATE OF THE REMAINDER OF SOME ALTERNATING SERIES

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Abstract. For any two real numbers $\alpha > 0$ and $\beta > -\alpha$, we show that the best constants a and b (the smallest a and the largest b) such that the inequalities

$$\frac{1}{2\alpha n + a} < \left| \sum_{k=n+1}^{\infty} \frac{(-1)^{k-1}}{\alpha k + \beta} \right| < \frac{1}{2\alpha n + b}$$

hold for every $n \geqslant 1$ are $a = \left(\frac{1}{\alpha + \beta} - S(\alpha,\beta) \right)^{-1} - 2\alpha$ and $b = \alpha + 2\beta$, where $S(\alpha,\beta) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\alpha n + \beta}$. In particular, we recover the main result of [6] and answer a question, stated in [6], about the Gregory-Leibniz series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$. More precisely, we show that the best constants c and d such that the inequalities

$$\frac{1}{4n + c} < \left| \sum_{k=n+1}^{\infty} \frac{(-1)^{k-1}}{2k-1} \right| < \frac{1}{4n + d}$$

hold for every $n \geqslant 1$ are $c = \frac{4}{4-\pi} - 4$ and $d = 0$.

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