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SHARP, DOUBLE INEQUALITIES BOUNDING THE FUNCTION $(1+x)^{1/x}$ AND A REFINEMENT OF CARLEMAN'S INEQUALITY

VITO LAMPRET

Abstract. In the expansion

$$(1+x)^{1/x} = e \cdot \sum_{j=0}^{\infty} (-1)^j B_j \cdot \left(\frac{x}{x+2}\right)^j, \quad \text{for } -1 < x \neq 0,$$

the sequence B_n is monotonically decreasing, bounded as $\frac{7}{10} < \lim_{n\to\infty} B_n < B_n < \frac{8}{10}$, for $n \geqslant 4$, and is given recursively as

$$B_0 = 1 \quad \text{and} \quad B_{2m} = B_{2m+1} = \frac{1}{m} \sum_{j=1}^{m} \frac{4j+1}{4j+2} B_{2m-2j}, \quad \text{for} \quad m \geqslant 1.$$

For any integers $m,n \geqslant 1$, the double inequality

$$e \cdot \sum_{j=0}^{2m-1} (-1)^j \frac{B_j}{(2n+1)^j} < \left(1+\frac{1}{n}\right)^n < e \cdot \sum_{j=0}^{2m} (-1)^j \frac{B_j}{(2n+1)^j}$$

holds, together with improved Carleman's inequality

$$\sum_{n=1}^{\infty} \left(\prod_{i=1}^{n} x_i\right)^{1/n} < e \cdot \sum_{n=1}^{\infty} \left(1 - \sum_{j=1}^{2m} (-1)^{j+1} \frac{B_j}{(2n+1)^j}\right) x_n,$$

true for every sequence $x_n \geqslant 0$ such that $0 < \sum_{n=1}^{\infty} x_n < \infty$.

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REFERENCES

[1] T. CARLEMAN, *Sur les fonctions quasi-analytiques*, Comptes rendus du V^e Congres des Mathematiciens Scandinaves, Helsingfors (1922), 181–196.
[2] H.-W. CHEN, *On an infinite series for $(1+1/x)^x$ and its application*, Int. J. Math. Math. Sci. **29** (2002), 675–680.
[3] C.-P. CHEN AND J. CHOI, *Asymptotic formula for $(1+1/x)^x$ based on the partition function*, Amer. Math. Monthly **121** (2014), 338–343.
[4] C.-P. CHEN AND H. J. ZHANG, *Padé approximant related to inequalities involving the constant e and a generalized Carleman-type inequality*, Journal of Inequalities and Applications, (2017) 2017: 205, doi:10.1186/s13660-017-1479-8.
[5] C.-P. CHEN AND R. B. PARIS, *An inequality involving the constant e and a generalized Carleman-type inequality*, J. Math. Anal. Appl. **466** (2018), 711–725.
[6] C.-P. CHEN AND R. B. PARIS, *An inequality involving the constant e and a generalized Carleman-type inequality*, Math. Inequal. Appl. **23** (2020), 1197–1203.

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[7] A. ČIŽMEŠIJA, J. PEČARIĆ AND L. E. PERSSON, *On strengtened weighted Carleman's inequality*, Bull. Austral. Math. Soc. **68** (2003), 481–490.
[8] G. H. HARDY, *Notes on some points in the integral calculus*, Messenger of Math. **54** (1925), 150–156.
[9] H.-P. LIU AND L. ZHU, *New strengthened Carleman's inequality and Hardy's inequality*, J. Inequal. and Appl. (2007), Art. ID 84104, 7 pp.
[10] C. MORTICI AND Y. HU, *On some convergences to the constant e and improvements of Carleman's inequality*, Carpathian J. Math. **31** (2015), 249–254.
[11] C. MORTICI AND X.-J. JANG, *Estimates of $(1+x)^{1/x}$ involved in Carleman's inequality and Keller's limit*, Filomat **29**, (2015), 1535–1539.
[12] G. PÓLYA, *Proof of an inequality*, Proc. London Math. Soc. **24** (1926), 57.
[13] V. PONOMARENKO, *Asymptotic formula for $(1+1/x)^x$, Revisited*, Amer. Math. Monthly **122** (2015), p. 587.
[14] Z. XIE AND Y. ZHONG, *A best approximation for constant e and an improvement to Hardy's inequality*, J. Math. Anal. Appl. **252** (2000), 994–998.
[15] S. WOLFRAM, *Mathematica*, **version 7.0**, Wolfram Research, Inc., 1988–2009.
[16] B. YANG, *On Hardy's inequality*, J. Math. Anal. Appl. **234** (1999), 717–722.
[17] X. YOU, DI-R. CHEN AND H. SHI, *Continued fraction inequalities related to $(1+1/x)^x$*, J. Math. Anal. Appl. **443** (2016) 1090–1094.