Mathematical
Inequalities
& Applications
Volume 26, Number 1 (2023), 161–182 doi:10.7153/mia-2023-26-13

INTEGRAL EQUATIONS ON COMPACT MANIFOLD WITH BOUNDARY

YAZHOU HAN

Abstract. Let (M^n, g, Σ) be a smooth compact Riemannian manifold with boundary and $n \geqslant 3$. This paper is devoted to studying a class of integral system

$$\begin{cases} g^{p_\alpha - 1}(x) = \int_\Sigma K(x,y) f(y) dS_y, & x \in M^n, \\ f^{\tilde{p}_\alpha - 1}(y) = \int_{M^n} K(x,y) g(x) dV_x, & y \in \Sigma, \end{cases}$$

where $\alpha \in (1,n)$, $p_\alpha = \frac{2n}{n+\alpha}$, $\tilde{p}_\alpha = \frac{2(n-1)}{n+\alpha-2}$, $(f,g) \in L^{\tilde{p}_\alpha}(\Sigma) \times L^{p_\alpha}(M^n)$ and the kernel function $K(x,y) \in C^\infty(\overline{M^n} \times \overline{M^n} \backslash \{(x,x)\})$ satisfies $K(x,y) \sim |x-y|_g^{\alpha-n}$ as $|x-y|_g \to 0$. Since the system is the Euler-Lagrange equations of extremal problem

$$N_K(\alpha, M) = \sup\left\{ \left| \int_{M^n} \int_\Sigma g(x) K(x,y) f(y) dS_y dV_x \right| : \|f\|_{L^{\tilde{p}_\alpha}(\Sigma)} = \|g\|_{L^{p_\alpha}(M^n)} = 1 \right\},$$

we will study the existence of the system by concentration-compactness principle. Firstly, we get $N_K(\alpha, M) \geqslant C_e(n, \alpha, \tilde{p}_\alpha)$, where $C_e(n, \alpha, \tilde{p}_\alpha)$ is the best constant of Hardy-Littlewood-Sobolev inequalities on the upper half space established by Dou and Zhu [6] and equals to $N_K(\alpha, M)$ when $(M^n, g, \Sigma) = (B_1(0), |\cdot|, \partial B_1(0))$ and $K(x,y) = |x-y|^{\alpha-n}$. Secondly, if $N_K(\alpha, M) > C_e(n, \alpha, \tilde{p}_\alpha)$, we prove that $N_K(\alpha, M)$ is attained. Namely, under the criterion $N_K(\alpha, M) > C_e(n, \alpha, \tilde{p}_\alpha)$, we get the existence of the system. Lastly, a concrete example satisfying the criterion is given. The example is closely related to the conformal problems studied by Escobar [9, 10].

Mathematics subject classification (2020): 45G15, 35A23, 45E10.

Keywords and phrases: Integral equations, concentration-compactness principle, compact Riemannian manifold with boundary, Hardy-Littlewood-Sobolev inequality.

REFERENCES

[1] S. CHEN, *A new family of sharp conformally invariant integral inequalities*, Int. Math. Res. Not., 5 (2014), 1205–1220.

[2] L. CHEN, G. LU, C. TAO, *Reverse Stein-Weiss inequalities on the upper half space and the existence of their extremals*, Adv. Nonlinear Stud., 19 (2019), 475–494.

[3] L. CHEN, G. LU, C. TAO, *Existence of extremal functions for the Stein-Weiss inequalities on the Heisenberg group*, J. Funct. Anal., 277 (2019), 1112–1138.

[4] J. DOU, Q. GUO, M. ZHU, *Subcritical approach to sharp Hardy-Littlewood-Sobolev type inequalities on the upper half space*, Adv. Math., 312 (2017), 1–45. Corrigendum to "Subcritical approach to sharp Hardy-Littlewood-Sobolev type inequalities on the upper half space" [Adv. Math. 312 (2017) 1–45], Adv. Math., 317 (2017), 640–644.

[5] J. DOU, Q. GUO, M. ZHU, *Negative power nonlinear integral equations on bounded domains*, J. Diff. Eqs., 269 (2020), 10527–10557.

[6] J. DOU, M. ZHU, *Sharp Hardy-Littlewood-Sobolev inequality on the upper half space*, Int. Math. Res. Not., 3 (2015), 651–687.

[7] J. DOU, M. ZHU, *Reversed Hardy-Littewood-Sobolev inequality*, Int. Math. Res. Not., 19 (2015), 9696–9728.

[8] J. DOU, M. ZHU, *Nonlinear integral equations on bounded domains*, J. Funct. Anal., 277 (2019), 111–134.

[9] J. F. ESCOBAR, *The Yamabe problem on manifolds with boundary*, J. Diff. Geom., 35 (1992), 21–84.

[10] J. F. ESCOBAR, *Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary*, Ann. of Math. 136 (1992), 1–50.

[11] R. L. FRANK, E. H. LIEB, *Sharp constants in several inequalities on the Heisenberg group*, Ann. of Math., 176 (2012), 349–381.

[12] M. GLUCK, *Subcritical approach to conformally invariant extension operators on the upper half space*, J. Funct. Anal., 278 (1) (2020), 1–46.

[13] M. GLUCK, M. ZHU, *An extension operator on bounded domains and applications*, Calc. Var. PDE, 58 (2019), Article No. 79.

[14] Y. HAN, *Integral equations on compact CR manifolds*, Discrete and Continuous Dynamical Systems, 41 (5) (2021), 2187–2204.

[15] Y. HAN, M. ZHU, *Hardy-Littlewood-Sobolev inequalities on compact Riemannian manifolds and applications*, J. Diff. Eqs., 260 (2016), 1–25.

[16] F. HANG, X. WANG, X. YAN, *Sharp integral inequalities for harmonic functions*, Comm. Pure Appl. Math., 61 (2008), 0054–0095.

[17] F. HANG, X. WANG, X. YAN, *An integral equation in conformal geometry*, Ann. Inst. H. Poincaré Analyse Non Linéaire, 26 (2009), 1–21.

[18] G. H. HARDY, J. E. LITTLEWOOD, *Some properties of fractional integrals (1)*, Math. Zeitschr., 27 (1928), 565–606.

[19] G. H. HARDY, J. E. LITTLEWOOD, *On certain inequalities connected with the calculus of variations*, J. London Math. Soc., 5 (1930), 34–39.

[20] D. JERISON, J. M. LEE, *The Yamabe problem on CR manifolds*, J. Diff. Geom. 25 (1987), 167–197.

[21] E. H. LIEB, *Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities*, Ann. of Math. 118 (1983), 349–374.

[22] P. L. LIONS, *The concentration-compactness principle in the calculus of variations. The limit case, Part 1*, Rev. Mat. Iberoamericana, 1 (1) (1985), 145–201.

[23] P. L. LIONS, *The concentration-compactness principle in the calculus of variations. The limit case, Part 2*, Rev. Mat. Iberoamericana, 1 (2) (1985), 45–121.

[24] Q. N. NGÔ, V. H. NGUYEN, *Sharp reversed Hardy-Littlewood-Sobolev inequality on the half space* $\mathbb{R}^n_+$, Int. Math. Res. Not., 20 (2017), 6187–6230.

[25] Q. N. NGÔ, *All conditions for Stein-Weiss inequalities are necessary*, arXiv: 2110.14220[math. FA].

[26] Q. N. NGÔ, Q. H. NGUYEN, V. H. NGUYEN, *An optimal Hardy-Littlewood-Sobolev inequality on* $\mathbb{R}^{n-k}\times\mathbb{R}^n$ *and its consequences*, arXiv: 2009.09868[math.FA].

[27] S. L. SOBOLEV, *On a theorem of functional analysis*, Mat. Sb. (N.S.), 4 (1938), 471–479. A. M. S. Transl. Ser., 2, 34 (1963), 39–68.

[28] M. ZHU, *Prescribing integral curvature equation*, Diff. Inte. Eqs., 29 (2016), 889–904.