

NEW ORDERS AMONG HILBERT SPACE OPERATORS

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Abstract. This article introduces several new relations among related Hilbert space operators. In particular, we prove some Löwner partial orderings among T , $|T|$, $\Re T$, $\Im T$, $|T| + |T^*|$ and many other related forms, as a new discussion in this field; where $\Re T$ and $\Im T$ are the real and imaginary parts of the operator T . Our approach will be based on proving the positivity of some new matrix operators, where several new forms for positive matrix operators will be presented as a key tool in obtaining the other ordering results. As an application, we present some results treating numerical radius inequalities in a way that extends some known results in this direction, in addition to some results about the singular values.

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REFERENCES

- [1] T. ANDO, *Geometric mean and norm Schwarz inequality*, Ann. Funct. Anal., **7**, 1 (2016), 1–8.
- [2] T. ANDO, *Topics on operator inequalities*, Hokkaido Univ. Lecture Note, 1978.
- [3] T. ANDO AND F. HIAI, *Operator log-convex functions and operator means*, Math. Ann., **350**, (2011), 611–630.
- [4] W. AUDEH AND F. KITTANEH, *Singular value inequalities for compact operators*, Linear Algebra Appl., **437**, (2012), 2516–2522.
- [5] Y. BEDRANI, F. KITTANEH, AND M. SABABHEH, *Numerical radii of accretive matrices*, Linear Multilinear Algebra, **69**, (2021), 957–970.
- [6] P. BHUNIA, S. BAG AND K. PAUL, *Numerical radius inequalities and its applications in estimation of zeros of polynomials*, Linear Algebra Appl., **573**, (2019), 166–177.
- [7] R. BHATIA, *Matrix analysis*, Springer, New York, 1997.
- [8] R. BHATIA, *Positive definite matrices*, Princeton Ser. Appl. Math., Princeton Univ. Press, Princeton, 2007.
- [9] R. BHATIA, F. KITTANEH, *Norm inequalities for positive operators*, Lett. Math. Phys., **43**, (1998), 225–231.
- [10] R. BHATIA AND F. KITTANEH, *On the singular values of a product of operators*, SIAM J. Matrix Anal. Appl., **11**, (1990), 272–277.
- [11] D. CHEN AND Y. ZHANG, *Singular value inequalities for real and imaginary parts of matrices*, Filomat, **30**, 10 (2016), 2623–2629.
- [12] S. S. DRAGOMIR AND M. S. MOSLEHIAN, *Some inequalities for (α, β) -normal operators in Hilbert spaces*, Facta Universitatis, **23**, (2008), 39–47.
- [13] K. FEKI AND T. YAMAZAKI, *Joint numerical radius of spherical Aluthge transforms of tuples of Hilbert space operators*, Math. Ineq. Appl., **24**, (2021), 405–420.
- [14] N. HIGHAM, *Functions of matrices: Theory and Computation*, SIAM, 2008.
- [15] O. HIRZALLAH AND F. KITTANEH, *Inequalities for sums and direct sums of Hilbert space operators*, Linear Algebra Appl., **424**, (2007), 71–82.
- [16] T. KATO, *Notes on some inequalities for linear operators*, Math. Ann., **125**, (1952), 208–212.
- [17] F. KITTANEH, *A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix*, Studia Math., **158**, 1 (2003), 11–17.

- [18] F. KITTANEH, *Notes on some inequalities for Hilbert space operators*, Publ. Res. Inst. Math. Sci., **24**, (1988), 283–293.
- [19] F. KITTANEH, *Numerical radius inequalities for Hilbert space operators*, Studia Math., **168**, 1 (2005), 73–80.
- [20] F. KUBO AND T. ANDO, *Means of positive linear operators*, Math. Ann., **246**, (1979/80), 205–224.
- [21] H. R. MORADI AND M. SABABHEH, *New estimates for the numerical radius*, Filomat, **35**, (2021), 4957–4962.
- [22] S. SHEYBANI, M. SABABHEH AND H. R. MORADI, *Weighted inequalities for the numerical radius*, Vietnam J. Math., **51** (2023), 363–377.
- [23] Y. TAO, *More results on singular value inequalities of matrices*, Linear Algebra Appl., **416**, (2006), 724–729.
- [24] D. XIA, *On the non-normal operators-semi-hyponormal operators*, Sci. Sinica., **23**, (1980), 700–713.
- [25] T. YAMAZAKI, *On upper and lower bounds of the numerical radius and an equality condition*, Stud. Math., **178**, (2007), 83–89.
- [26] A. ZAMANI, *Numerical radius in Hilbert C^* -modules*, Math. Ineq. Appl., **24**, (2021), 1017–1030.
- [27] X. ZHAN, *Matrix inequalities*, Springer-Verlag, Berlin, 2002.