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AN IMPROVED SPIRA'S INEQUALITY FOR THE RIEMANN ZETA FUNCTION AND ITS DERIVATIVES IN THE CRITICAL STRIP

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Abstract. The region of validity of Spira's strict inequality, given by $|\zeta(1-s)| = g(s)|\zeta(s)|$ where $g(s) := 2^{1-s}\pi^{-s}\cos(\pi s/2)\Gamma(s)$, with $g(s) > 1$, involving the size of the Riemann zeta-function, $\zeta(s)$, at places symmetric with respect to the critical line, is enlarged to the subset $H_{t_*} := H \cap \{t > t_*\}$ of the semi-infinite critical half-strip $H := \{(\sigma,t) \in \mathbf{C} : 1/2 < \sigma < 1, t > 0\}$, where $s = \sigma + it$ and $t_* = 2\pi + \varepsilon = 6.380685^+$. It is conjectured that a smooth line, ℓ, exists in H such that the Spira's inequality holds above ℓ, while the opposite inequality holds below ℓ, and equality holds on ℓ. Moreover, if a nontrivial zero, s_0, of $\zeta(s)$ of multiplicity k exists in H_{t_*}, it is shown that $|\zeta^{(k)}(1-s_0)| > |\zeta^{(k)}(s_0)|$.

Mathematics subject classification (2020): 11M06, 11M99, 11M26.
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