

A LOGARITHMICALLY COMPLETELY MONOTONIC FUNCTION AND SEVERAL INEQUALITIES FOR q -MULTINOMIAL COEFFICIENTS AND q -MULTIVARIATE BETA FUNCTIONS

FENG QI

Abstract. In the paper, the author proves the logarithmically complete monotonicity of a function involving q -gamma functions and apply the logarithmically complete monotonicity to derive some inequalities for q -multinomial coefficients and q -multivariate beta functions. These conclusions generalize corresponding ones for gamma functions, multinomial coefficients, and multivariate beta functions, respectively.

Mathematics subject classification (2020): 05A20, 26A48, 26D07, 33B15, 44A10.

Keywords and phrases: Logarithmically complete monotonicity, combinatorial inequality, q -gamma function, q -multinomial coefficient, q -multivariate beta function.

REFERENCES

- [1] M. ABRAMOWITZ AND I. A. STEGUN (Eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series **55**, 10th printing, Dover Publications, New York and Washington, 1972.
- [2] H. ALZER, *Complete monotonicity of a function related to the binomial probability*, J. Math. Anal. Appl. **459** (2018), no. 1, 10–15, <https://doi.org/10.1016/j.jmaa.2017.10.077>.
- [3] G. E. ANDREWS, R. ASKEY, AND R. ROY, *Special Functions*, Encyclopedia of Mathematics and its Applications **71**, Cambridge University Press, Cambridge, 1999, <http://dx.doi.org/10.1017/CB09781107325937>.
- [4] C. BERG, *Integral representation of some functions related to the gamma function*, Mediterr. J. Math. **1** (2004), no. 4, 433–439, <http://dx.doi.org/10.1007/s00009-004-0022-6>.
- [5] G. GASPER AND M. RAHMAN, *Basic Hypergeometric Series*, 2nd ed., Encyclopedia of Mathematics and its Applications **96**, Cambridge University Press, Cambridge, 2004, <http://dx.doi.org/10.1017/CB09780511526251>.
- [6] B.-N. GUO AND F. QI, *A completely monotonic function involving the tri-gamma function and with degree one*, Appl. Math. Comput. **218** (2012), no. 19, 9890–9897, <http://dx.doi.org/10.1016/j.amc.2012.03.075>.
- [7] B.-N. GUO AND F. QI, *A property of logarithmically absolutely monotonic functions and the logarithmically complete monotonicity of a power-exponential function*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **72** (2010), no. 2, 21–30.
- [8] B.-N. GUO AND F. QI, *On the degree of the weighted geometric mean as a complete Bernstein function*, Afr. Mat. **26** (2015), no. 7, 1253–1262, <http://dx.doi.org/10.1007/s13370-014-0279-2>.
- [9] B.-N. GUO AND F. QI, *Properties and applications of a function involving exponential functions*, Commun. Pure Appl. Anal. **8** (2009), no. 4, 1231–1249, <http://dx.doi.org/10.3934/cpaa.2009.8.1231>.
- [10] M. E. H. ISMAIL AND M. E. MULDOON, *Inequalities and monotonicity properties for gamma and q -gamma functions*, in: R.V.M. Zahar (Ed.), *Approximation and Computation: A Festschrift in Honour of Walter Gautschi*, ISNM, vol. **119**, BirkhRauser, Basel, 1994, 309–323, http://dx.doi.org/10.1007/978-1-4684-7415-2_19.

- [11] M. E. H. ISMAIL AND M. E. MULDOON, *Inequalities and monotonicity properties for gamma and q -gamma functions*, arXiv (2013), available online at <http://arxiv.org/abs/1301.1749>.
- [12] E. KOELINK AND W. VAN ASSCHE, *Leonhard Euler and a q -analogue of the logarithm*, Proc. Amer. Math. Soc. **137** (2009), no. 5, 1663–1676, <https://doi.org/10.1090/S0002-9939-08-09374-X>.
- [13] A. LEBLANC AND B. C. JOHNSON, *On a uniformly integrable family of polynomials defined on the unit interval*, J. Inequal. Pure Appl. Math. **8** (2007), no. 3, Article 67, 5 pp. Available online at <https://www.emis.de/journals/JIPAM/article878.html>.
- [14] A. W. MARSHALL, I. OLKIN, AND B. C. ARNOLD, *Inequalities: Theory of Majorization and its Applications*, 2nd ed., Springer Verlag, New York-Dordrecht-Heidelberg-London, 2011, <http://dx.doi.org/10.1007/978-0-387-68276-1>.
- [15] D. S. MITRINOVIĆ, J. E. PEČARIĆ, AND A. M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht-Boston-London, 1993, <http://dx.doi.org/10.1007/978-94-017-1043-5>.
- [16] F. W. J. OLVER, D. W. LOZIER, R. F. BOISVERT, AND C. W. CLARK (Eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, New York, 2010, <http://dlmf.nist.gov/>.
- [17] F. OUIMET, *Complete monotonicity of multinomial probabilities and its application to Bernstein estimators on the simplex*, J. Math. Anal. Appl. **466** (2018), no. 2, 1609–1617, <https://doi.org/10.1016/j.jmaa.2018.06.049>.
- [18] F. OUIMET AND F. QI, *Logarithmically complete monotonicity of a matrix-parametrized analogue of the multinomial distribution*, Math. Inequal. Appl. **25** (2022), no. 3, 703–714, <http://dx.doi.org/10.7153/mia-2022-25-45>.
- [19] F. QI, *A completely monotonic function related to the q -trigamma function*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **76** (2014), no. 1, 107–114.
- [20] F. QI, *A logarithmically completely monotonic function involving the q -gamma function*, HAL preprint (2018), available online at <https://hal.archives-ouvertes.fr/hal-01803352v1>.
- [21] F. QI, *Bounds for the ratio of two gamma functions*, J. Inequal. Appl. **2010**, Article ID 493058, 84 pages, <http://dx.doi.org/10.1155/2010/493058>.
- [22] F. QI, *Certain logarithmically N -alternating monotonic functions involving gamma and q -gamma functions*, Nonlinear Funct. Anal. Appl. **12** (2007), no. 4, 675–685.
- [23] F. QI, *Complete monotonicity for a new ratio of finitely many gamma functions*, Acta Math. Sci. Ser. B (Engl. Ed.) **42B** (2022), no. 2, 511–520, <https://doi.org/10.1007/s10473-022-0206-9>.
- [24] F. QI, *Complete monotonicity of functions involving the q -trigamma and q -tetragamma functions*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **109** (2015), no. 2, 419–429, <http://dx.doi.org/10.1007/s13398-014-0193-3>.
- [25] F. QI, *Limit formulas for ratios between derivatives of the gamma and digamma functions at their singularities*, Filomat **27** (2013), no. 4, 601–604, <http://dx.doi.org/10.2298/FIL1304601Q>.
- [26] F. QI, *Properties of modified Bessel functions and completely monotonic degrees of differences between exponential and trigamma functions*, Math. Inequal. Appl. **18** (2015), no. 2, 493–518, <http://dx.doi.org/10.7153/mia-18-37>.
- [27] F. QI AND R. P. AGARWAL, *On complete monotonicity for several classes of functions related to ratios of gamma functions*, J. Inequal. Appl. **2019**, Paper No. 36, 42 pages, <https://doi.org/10.1186/s13660-019-1976-z>.
- [28] F. QI AND C.-P. CHEN, *A complete monotonicity property of the gamma function*, J. Math. Anal. Appl. **296** (2004), 603–607, <http://dx.doi.org/10.1016/j.jmaa.2004.04.026>.
- [29] F. QI AND B.-N. GUO, *From inequalities involving exponential functions and sums to logarithmically complete monotonicity of ratios of gamma functions*, J. Math. Anal. Appl. **493** (2021), no. 1, Article 124478, 19 pages, <https://doi.org/10.1016/j.jmaa.2020.124478>.
- [30] F. QI, B.-N. GUO, AND C.-P. CHEN, *Some completely monotonic functions involving the gamma and polygamma functions*, J. Aust. Math. Soc. **80** (2006), 81–88, <http://dx.doi.org/10.1017/S1446788700011393>.
- [31] F. QI AND W.-H. LI, *Integral representations and properties of some functions involving the logarithmic function*, Filomat **30** (2016), no. 7, 1659–1674, <https://doi.org/10.2298/FIL1607659Q>.
- [32] F. QI, W.-H. LI, S.-B. YU, X.-Y. DU, AND B.-N. GUO, *A ratio of finitely many gamma functions and its properties with applications*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM. **115** (2021), no. 2, Paper No. 39, 14 pages, <https://doi.org/10.1007/s13398-020-00988-z>.

- [33] F. QI AND D. LIM, *Monotonicity properties for a ratio of finite many gamma functions*, Adv. Difference Equ. **2020**, Paper No. 193, 9 pages, <https://doi.org/10.1186/s13662-020-02655-4>.
- [34] F. QI AND A.-Q. LIU, *Completely monotonic degrees for a difference between the logarithmic and psi functions*, J. Comput. Appl. Math. **361** (2019), 366–371, <https://doi.org/10.1016/j.cam.2019.05.001>.
- [35] F. QI, F.-F. LIU, AND X.-T. SHI, *Comments on two completely monotonic functions involving the q -trigamma function*, J. Inequal. Spec. Funct. **7** (2016), no. 4, 211–217.
- [36] F. QI AND Q.-M. LUO, *Bounds for the ratio of two gamma functions – From Wendel’s and related inequalities to logarithmically completely monotonic functions*, Banach J. Math. Anal. **6** (2012), no. 2, 132–158, <http://dx.doi.org/10.15352/bjma/1342210165>.
- [37] F. QI, D.-W. NIU, D. LIM, AND B.-N. GUO, *Some logarithmically completely monotonic functions and inequalities for multinomial coefficients and multivariate beta functions*, Appl. Anal. Discrete Math. **14** (2020), no. 2, 512–527, <https://doi.org/10.2298/AADM191111033Q>.
- [38] F. QI AND S.-H. WANG, *Complete monotonicity, completely monotonic degree, integral representations, and an inequality related to the exponential, trigamma, and modified Bessel functions*, Glob. J. Math. Anal. **2** (2014), no. 3, 91–97, <http://dx.doi.org/10.14419/gjma.v2i3.2919>.
- [39] F. QI, X.-J. ZHANG, AND W.-H. LI, *Lévy-Khintchine representations of the weighted geometric mean and the logarithmic mean*, Mediterr. J. Math. **11** (2014), no. 2, 315–327, <http://dx.doi.org/10.1007/s00009-013-0311-z>.
- [40] R. L. SCHILLING, R. SONG, AND Z. VONDRAČEK, *Bernstein Functions – Theory and Applications*, 2nd ed., de Gruyter Studies in Mathematics **37**, Walter de Gruyter, Berlin, Germany, 2012, <http://dx.doi.org/10.1515/9783110269338>.
- [41] D. V. WIDDER, *The Laplace Transform*, Princeton University Press, Princeton, 1946.