

## EMBEDDINGS AND RELATED TOPICS IN GRAND VARIABLE EXPONENT HAJŁASZ–MORREY–SOBOLEV SPACES

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*Abstract.* Embeddings in the framework of grand variable exponent function spaces are studied. In particular, embeddings from grand variable exponent Hajłasz-Sobolev-Morrey spaces to variable exponent Hölder spaces are established. The regularity of a fractional integral operator defined with respect to a non-doubling measure is also investigated. In particular, mapping properties of this operator from a grand variable exponent Morrey space to a grand variable parameter Hölder space are studied. The results are proved under the log-Hölder continuity condition on the exponents. The spaces are defined, generally speaking, on quasi-metric measure spaces, however, the results are new even for Euclidean spaces.

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